WHAT IS...

an Alteration?

Frans Oort

Motivation. Algebraic varieties, which are zero sets of polynomial equations, appear in geometry, in number theory, and in analysis. For example, Fermat's Last Theorem studies rational solutions of a polynomial equation. The Weierstrass $\wp$-function and its derivative satisfy a cubic polynomial equation. The geometry of these varieties gives important information on the related problems in number theory or analysis.

In a variety the smooth locus is dense, but there can be singularities, such as a node, a cusp on an algebraic curve, a "self-intersection" on a surface, a quotient singularity, and much more complicated forms. Often singularities make it difficult to carry out proofs, to analyze properties of the problem described by the variety, to compute integrals, and so on.

Therefore it is natural to ask: *Is it possible to "desingularize" a given algebraic variety, to replace it in a reasonable way by a nonsingular variety?*

This question was studied and solved successfully for curves and for algebraic surfaces. Oscar Zariski was one of the main contributors, and he stimulated many researchers in this problem. We should pay a tribute to him!

Examples. a) A typical example: let C be the plane curve given by the equation $X^3 = Y^2$. At $(x = 0, y = 0)$ this curve has a singularity, a *cusp*. We map affine space of dimension one by $u \mapsto (u^2, u^3)$ onto C; this is a resolution of singularities of the singular curve C.

b) Consider the plane curve given by the equation $X^4 + Y^4 = 1$, the *Fermat quartic*. Fermat's Last Theorem tells us (as Fermat proved himself in this special case) that there are no rational solutions $(x, y) \in \mathbb{Q} \times \mathbb{Q}$ to this equation with $xy \neq 0$.

c) Consider the plane curve, the *Bernoulli lemniscate*, given by the equation $(X^2 + Y^2)^2 = X^2 - Y^2$. In this case there are infinitely many rational solutions. One might have difficulties finding them without analyzing the geometry of the related projective curve.

d) Consider a nonsingular variety V, e.g. affine space $\mathbb{C}^n$, and consider a finite group G acting on it. In general the quotient space has singularities, the so-called *quotient singularities*. For example, let $V = \mathbb{C}^2$, and let G be the cyclic group of order two, generated by reflection in the origin. In this case $G\backslash V$ is a surface, and the only singularity is the image of the point $(0, 0)$.

Modifications: Resolution of Singularities. We make more precise what we mean by "changing a variety into a nonsingular one".

Definition. *A morphism $f : W \to V$ of algebraic varieties is called a* modification *if it is birational and proper.*

"Birational" means that f is an isomorphism outside lower-dimensional subvarieties, i.e., an isomorphism almost everywhere. "Proper" means that f is a closed map. The reader may have encountered analogous terminology or related concepts such as a blowing-up, a σ-process, or a Cremona transformation.

Frans Oort is emeritus professor of mathematics at the Universiteit Utrecht. His email address is `oort@math.uu.nl`.

The map in example (a) is a modification. The quotient map $V \to G\backslash V$ is not a modification if the action of G on V is nontrivial.

Definition. *A morphism $f : W \to V$ is called a* resolution of singularities *of V if f is a modification and W is a nonsingular variety.*

Hironaka's Theorem. In 1964 Hironaka published his famous theorem on the resolution of singularities [3].

Theorem. *Let k be an algebraically closed field of* characteristic zero, *and let V be an algebraic variety over k. There exists a modification $f : W \to V$, where W is nonsingular.*

This useful theorem was a great breakthrough and admitted many applications. Hironaka's difficult construction has been carefully analyzed, simplified somewhat, and made canonical (see [2] for a survey). Moreover, in the theorem by Hironaka, we can choose f in such a way that it is an isomorphism exactly on the nonsingular part of V; this extra information has technical advantages.

Hironaka's proof, even in later versions, is difficult. Here is a challenge: try to resolve a quotient singularity, as in example (d).

Open Problem. Up to now we have not been able to prove the analogous theorem in positive characteristic (except for low-dimensional cases). There seems to be a need for a completely different approach [1]!

Alterations, A. J. de Jong, 1996.

Definition. *A morphism $f : W \to V$ of algebraic varieties is called an* alteration *if it is surjective, generically finite, and proper.*

Such a morphism is a closed map which is "almost everywhere" finite-to-one.

Theorem. *Let K be a field, and let V be an algebraic variety over K. There exists an* alteration *$f : W \to V$, where W is a nonsingular variety.*

From this theorem we can easily reprove Hironaka's theorem in a weak form (Bogomolov and Pantev, Abramovich and de Jong); for references and a discussion see the first chapter in [2].

Remark. On the one hand, every modification is an alteration. On the other hand, an *alteration* $f : W \to V$ can be factored, using the Stein factorization, as $W \to S \to V$, where $g : W \to S$ is a *modification* and $h : S \to V$ is a *finite morphism*. Any alteration (with V normal) where the corresponding h is not the identity is not a modification. The quotient map $\mathbb{C}^2 \to G\backslash\mathbb{C}^2$ in (d) is an alteration but not a modification.

A morphism of projective varieties is a modification if and only if it is birational (if and only if it is an isomorphism almost everywhere); it is an alteration if it is finite almost everywhere.

Sketch of the Proof of de Jong. In contrast with Hironaka's theorem, where the proof is difficult, the proof by de Jong is very clear. Basically (up to some technical difficulties) one replaces a (singular) variety V by a "fibration" $V \to B$, where all fibers are curves. By induction on the dimension, we assume the theorem to be proven for B: we pull back the fibration $V \to B$ via an alteration $B' \to B$, with B' nonsingular, to a fibration $V' \to B'$ over a nonsingular base. Resolution of singularities of curves is well known, and this can be made effective in the family $V' \to B'$.

Comparison: Modifications ↔ Alterations. In the proof by Hironaka, from the beginning we focus on singularities present. Explicit (algebraic) methods make singularities at one point "less singular" and, as Hironaka proves, do not make a "global invariant" worse. A complicated induction process gives the desired (deep, very useful) result.

In the method by de Jong, in the beginning of the proof singularities are completely ignored, and perhaps even more singularities are created. Explicit resolution of singularities is then carried out in the last, not very difficult, step. This geometric approach lends itself to many geometric situations. Instead of the algebraic, algorithmic approach by Hironaka, de Jong proposes a geometric method, which also works in relative situations (singularities in a family) and in positive characteristic. This allows several applications not possible in Hironaka's theory.

References

[1] A. J. de Jong, Smoothness, semi-stability and alterations, *Publ. Math. IHÉS* **83** (1996), 51–93. In this paper Johan de Jong introduces "alterations", and a proof is given that any variety is dominated via an alteration by a nonsingular variety.

[2] H. Hauser, J. Lipman, F. Oort, and A. Quirós, eds., *Resolution of Singularities. A Research Textbook in Tribute to Oscar Zariski*, Progr. Math., vol. 181, Birkhäuser-Verlag, Basel, 2000. Based on courses at a conference in Obergurgl, Austria, September 7–14, 1997. In this book we find many references, explanations, and full expositions on the techniques introduced by Hironaka and by de Jong.

[3] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, *Ann. Math.* **79** (1964), 109–326. In this paper Hironaka proves his famous theorem.

Comments and suggestions may be sent to `whatis-notices@ams.org`.