Indra's Pearls: The Vision of Felix Klein

Reviewed by Albert Marden

Indra's Pearls: The Vision of Felix Klein
David Mumford, Caroline Series, David Wright
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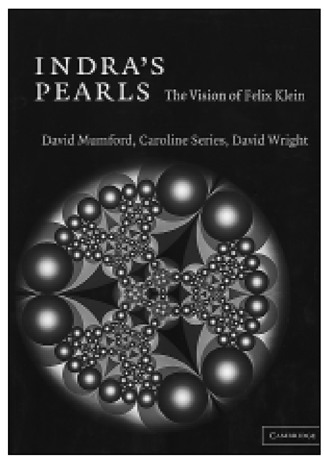
It has been a great pleasure to read such a gracefully written, original book of mathematics. Ignoring the mathematics, one can enjoy *Indra's Pearls* as an art book displaying deeply hidden fractal shapes—magical, mysterious, and beautiful.

The authors introduce from scratch a range of topics relevant to the central theme, including

- complex numbers,
- groups of symmetries,
- Möbius transformations,
- combinatorics and number theory of the modular group,
- geometric group theory,
- fractal geometry.

In a nutshell, the book is a walkabout in the space of two-generator Schottky groups and their various degenerations. Discrete groups of Möbius transformations, in particular Schottky groups, act not only on the extended complex plane but on upper half 3-space as well. A discrete group is always associated with a 3-manifold (or orbifold) and is often in addition associated with a surface or surfaces that form the boundary of the 3-manifold. The action



of most groups cannot be fully understood without involving the associated 3-manifolds. Yet it is completely appropriate that in their presentation, the authors have stuck to the group action on $\mathbb{S}^2$; to have done otherwise would have opened a Pandora's box which would have been inconsistent with the elementary nature of the exposition. However, I will describe their work from the more general perspective. I will also round out some of the historical references.

A two-generator Schottky group is associated with a surface of genus two that bounds a handlebody. If one imagines a surface of genus two smoothly embedded in $\mathbb{R}^3$, the region it encloses is called a handlebody of genus two. In fact, two-generator Schottky groups are perhaps the simplest class of "nonelementary" discrete groups. Yet typical phenomena that occur when working with more complicated groups already appear in the two-generator Schottky case and, because of the low dimension, can be fully exhibited on the computer monitor. That is why this is the perfect class to explore visually. The most interesting phenomena occur after the groups are geometrically "degenerated" in certain ways.

Indra's Pearls is a tale of Möbius transformations,

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$$A: z \mapsto \frac{az+b}{cz+d}, \quad ad - bc \neq 0.$$

The Möbius transformations comprise the class of conformal mappings of $\mathbb{C} \cup \infty \equiv \mathbb{S}^2$ onto itself. Each transformation A is associated with a matrix

$$A = \pm \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}),$$

normalized so that $ad - bc = 1$. The complex number $a + d$ (defined up to the factor ± 1) is called the *trace of* A.

A Möbius transformation is the composition of an even number of reflections in circles and lines. Being such, it has a natural extension to upper half 3-space. Upper halfspace, endowed with its hyperbolic metric, is a model of hyperbolic 3-space $\mathbb{H}^3$. The totality of extensions form the full group of orientation-preserving isometries of $\mathbb{H}^3$. The subgroup of Möbius transformations that in addition preserve the upper halfplane comprise the group of orientation-preserving isometries of hyperbolic 2-space $\mathbb{H}^2$.

A Möbius transformation, other than the identity, is classified as *elliptic* if conjugate to $z \mapsto e^{i\varphi} z$ with $\varphi \not\equiv 2\pi$, *parabolic* if conjugate to $z \mapsto z + 1$, and *loxodromic* if conjugate to $z \mapsto k e^{i\varphi} z$, $k > 1$. It is those loxodromics with $\varphi \not\equiv 0$ that are responsible for the spirals that are so decorative in the limit sets (see Figure 2).

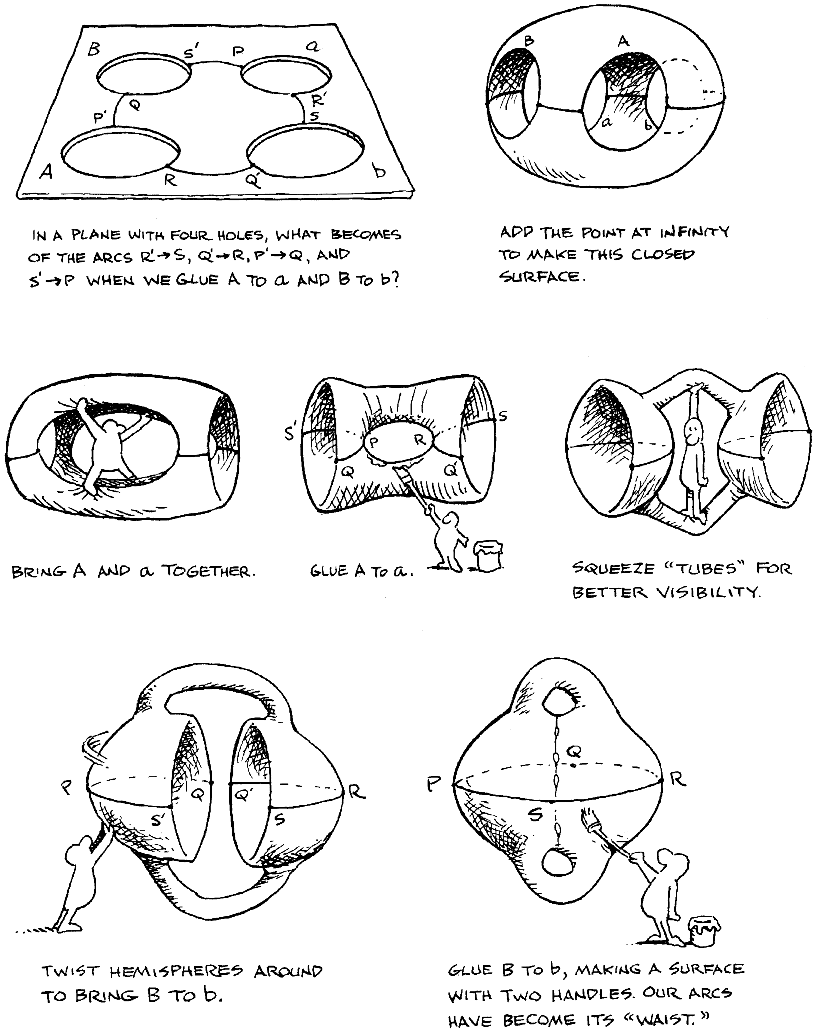

Figure 1. Dr. Stickler is gluing together each pair of Schottky circles so that their common exterior becomes a surface of genus 2 (Fig. 6.15 in the book). Here A, $a = A^{-1}$, B, $b = B^{-1}$ are the generators.

A discrete group G of Möbius transformations is called a *kleinian group*. If in addition it preserves the upper halfplane (or any disk in $\mathbb{S}^2$), it is called a *fuchsian group*. The *limit set* $\Lambda(G)$ is the set of accumulation points of the orbit of any point O in upper halfspace; the limit set lies in $\mathbb{S}^2$. The group G is called *elementary* if it is a finite group or if $\Lambda(G)$ consists of one or two points. For nonelementary groups, as all the groups we consider will be, the limit set is a closed perfect set in which the loxodromic fixed points are dense. The parabolic fixed points, if any, are dense as well. The complement $\Omega(G) = \mathbb{S}^2 \setminus \Lambda(G)$ is called the *set of discontinuity* or *regular set*. It is the largest open set in $\mathbb{S}^2$ in which G is properly discontinuous.

A two-generator Schottky group is a group G that arises from four mutually disjoint circles, $\{C_1, C_1'; C_2, C_2'\}$, bounding mutually disjoint disks in $\mathbb{C}$. Choose any Möbius transformation that sends the interior of C_i onto the exterior of its partner C_i', $i = 1, 2$, and label the pair A, B. The group $G = \langle A, B \rangle$ so generated is discrete and free. Its limit set $\Lambda(G)$ is totally disconnected with positive Hausdorff dimension and zero area ("fractal dust"). It corresponds to the "boundary" of the Cayley graph of G: each limit point is the limit of a sequence of nested circles. Therefore each limit point corresponds to a unique infinite word in four letters: the two generators plus their inverses. The common exterior of the four circles is a fundamental tile for the action of G on $\Omega(G)$.

Actually the group G so constructed is nowadays called a *classical Schottky group*. This is to distinguish it from the geometrically similar groups where the Schottky circles C_i no longer exist but are replaced by noncircular Jordan curves. Recent work of Hidalgo-Maskit has clarified how this distinction arises.

The quotient $S(G) = \Omega(G)/G$ is a Riemann surface of genus two (see Figure 1); the quotient $\mathcal{M}(G) = (\Omega(G) \cup \mathbb{H}^3)/G$ is a handlebody of genus two, as is a pretzel. Its interior $\mathbb{H}^3/G$ is complete in the projected hyperbolic 3-metric.

To begin the process of degeneration, find a simple loop $\gamma \subset S(G)$ that divides the surface into two tori, each with one boundary component, yet does *not* bound a topological disk within $\mathcal{M}(G)$.[1] "Pinch" the surface $S(G)$ along γ. This process yields a sequence of isomorphisms onto Schottky groups $\{\theta_n : G \to G_n\}$. The isomorphisms converge to an isomorphism $\theta : G \to G^*$ in the sense that for each generator $A, B \in G$, $\lim \theta_n(A) = \theta(A) = A^*$ exists, and likewise $\theta(B) = B^*$, while G^* is the group generated by these two limits. Such convergence, that is, convergence of generators, is called *algebraic convergence.* If $g \in G$ is an element corresponding to γ, then $\theta(g)$ is parabolic. This means that the commutator $[A^*, B^*]$ is parabolic if A, B are suitably chosen.

In the Hausdorff topology, $\lim \Lambda(G_n) = \Lambda(G^*)$. Instead of one infinitely connected component, $\Omega(G^*)$ has two simply connected components, $\Omega_{top}(G^*)$, $\Omega_{bot}(G^*)$. Each is invariant under the full group G^*. Their quotients, $S_{top}(G^*) = \Omega_{top}(G^*)/G^*$, $S_{bot}(G^*) = \Omega_{bot}(G^*)/G^*$, are once-punctured tori. To complete the picture, the quotient hyperbolic 3-manifold is a product $\mathcal{M}(G^*) \cong S_{bot}(G^*) \times [0, 1]$. Customarily, one of the components of $\partial\mathcal{M}(G^*)$ is referred to as the "top" and the other, the "bottom", as has already been suggested.

With this first pinch we arrive at the class of pinched Schottky groups of the type of G^*. Reverting to the original notation, this is the class of two generator, free, discrete groups of the form $G = \langle A, B \rangle$, where A, B are loxodromic but their commutator $[A, B] = ABA^{-1}B^{-1}$ is parabolic, and $\Omega(G)$ has two simply connected components, each invariant under G. Such a group is called a *quasifuchsian group* because it is the image of a fuchsian group under a quasiconformal mapping. The collection $\mathcal{QF}$ of such groups, modulo conjugation, comprises the *quasifuchsian once-punctured torus space.* The groups depend on two complex parameters that may be taken essentially as the traces of A and B; correspondingly, $\mathcal{QF}$ is a 2-dimensional complex manifold.

In the book this new class first appears by having the four Schottky circles become tangent—or, as the authors say, "kiss"—so as to form a circular quadrilateral. Each pairing transformation A, B must be chosen so as to fix the appropriate point of tangency. The result is called a "kissing Schottky group", and the resultant geometric picture in $\mathbb{C}$ is called "Indra's necklace" (Figures 6.16, 6.1). It is shown how the tangent circles give rise to a tiling of the two domains of discontinuity. In parallel, the authors explain how the "boundary" of the Cayley graph and the limit set congeal to become topological circles. This is because the initial loxodromic commutator becomes parabolic in the limit, fusing two fixed points into one.

Quasifuchsian space $\mathcal{QF}$ is naturally embedded as an open, connected, proper subset of the representation variety $\mathcal{R}(G)$ in $\mathrm{PSL}(2, \mathbb{C})$, modulo conjugation, where $\mathcal{R}(G)$ is formed subject to the restriction that the commutator must remain parabolic. As such, $\mathcal{QF}$ has a relative boundary $\partial\mathcal{QF}$. There are a countable number of special points on the boundary called *cusps.* These correspond to groups with new parabolics and, more particularly, are of the type of H_1 and H_2, to be introduced below.

By the mid-1970s, Troels Jorgensen had worked out a complete picture of $\mathcal{QF}$. He showed that the space can be described in terms of the combinatorics of the faces of the isometric (Ford) fundamental polyhedron for each group. To date, most of his work has not been published, yet it has become widely known and successfully applied, most recently by Makato Sakuma and colleagues to problems concerning 2-bridge knots. Jorgensen gave an alternate description of $\mathcal{QF}$ in terms of a "triangle graph": Each vertex represents a generator, and two generators represent the same vertex if and only if they are equal or inverses, modulo conjugations. Each edge represents a generator pair. The traces of the three vertices of a triangle are related by the Markoff identity. Jorgensen represented his triangle graph in the modular tessellation[2] so that the vertices can be indexed by the Farey series and continued fractions. Jorgensen proved that the Ford polyhedron $\mathcal{P}$ meets each component $\Omega_{top}, \Omega_{bot}$ in a circular polygon. He associated each of these two polygons with a triangle in his graph and then chose the minimal triangle strip in the graph joining the two triangles. Amazingly, the labeling of this strip not only describes the combinatorics of the faces of $\mathcal{P}$ but also reveals the group element associated with each face. Singly or doubly degenerate groups correspond to half-infinite or infinite geodesic strips. In this theory, the possible geometric limits[3] at a

[1] *Much later, in Chapter 8 (see Figure 8.20), an entirely different degeneration is introduced (called a Riley group): the two generators A, B become parabolic. The corresponding 3-manifold is the connected sum of two pinched solid tori: glue together two bagels with defective holes.*

[2] *The reference is to the famous tiling of the upper half-plane by the orbit of two adjacent ideal triangles under the level 2 congruence subgroup of the modular group, itself a pinched Schottky group, and the Farey series description of its ideal vertices.*

[3] *The sequences can be chosen to converge not only algebraically in the generators to a cusp group G^* but also* geometrically *in that the quotient manifolds converge to the manifold of a Kleinian group $H \supset G^*$. H will be strictly larger if the algebraic convergence to G^* is "tangential".*

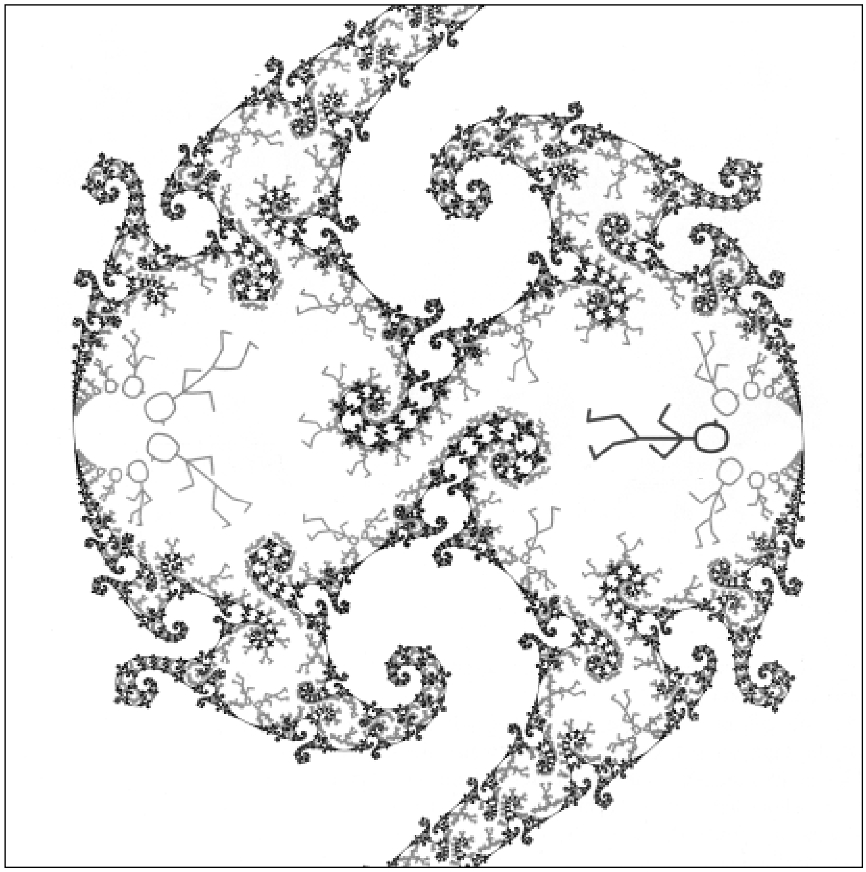

Figure 2. A blow-up of part of the limit set of a once-punctured torus quasifuchsian group. Dr. Stickler is shown being carried around one component of the regular set by the group action (Fig. 8.6 in the book).

cusp, including nonfinitely generated groups, can be deduced from the finite strip representing the cusp.

In particular, Jorgensen recognized that cyclic extensions H^* of certain of his boundary groups H—doubly degenerate groups corresponding to periodic infinite geodesic strips—give rise to manifolds $\mathcal{M}(H^*)$ that are fibered over the circle, with fibers being once-punctured tori. The discovery was a surprise; some had doubted the existence of such hyperbolic manifolds. For one of these groups,[4] H_0, $\mathcal{M}(H^*)$ is homeomorphic to the figure-8 knot complement. This is a consequence of the fact that H^* is conjugate to the group Bob Riley had earlier discovered to be a representation of the figure-8 knot group; this was one of the knot and link groups for which he had found representations in $\mathrm{PSL}(2,\mathbb{C})$. The group H_0 is the doubly degenerate example presented in the book.

Understanding of $\mathcal{QF}$ (and of deformation spaces for general groups) has dramatically increased in recent years. The manifold interiors $\mathbb{H}^3/H$ corresponding to all boundary groups $H \in \partial\mathcal{QF}$, as is the case for groups of $\mathcal{QF}$, are topological products $S \times (0,1)$, where S is a once-punctured torus (Thurston, Bonahon). McMullen showed that the limit set of any $H \in \partial\mathcal{QF}$ is locally connected. McMullen, Canary-Hersonsky proved that cusps are dense there. It follows from the ending lamination conjecture for $\mathcal{QF}$—proved by Yair Minsky, with additional results of Brock, Bromberg, Canary, and Minsky—that any group that "should be" on $\partial\mathcal{QF}$ by virtue of the product topology of its quotient actually is.

We can now do some more pinching. Take any simple loop γ_b in, say, $S_{bot}(G)$ not contractible to a point or a puncture, and "pinch" again. We get an algebraically convergent sequence of quasifuchsian groups, ending up with a group H_1 with the following property. In the limit, the top surface $S_{top}(H_1)$ remains a once-punctured torus. However, $S_{bot}(H_1)$ has become *the* 3-punctured sphere.[5]

Up in $\mathbb{S}^2$ there is still a component $\Omega_{top}(H_1)$ of $\Omega(H_1)$ that is simply connected and invariant under the full group H_1. In place of Ω_{bot} there is a countable union of disks, each preserved by a subgroup of H_1 conjugate to the famous level-two congruence subgroup of the modular group.

The cusp group H_1 itself has a 1-dimensional complex deformation space $\mathcal{QF}_1$, called a *Maskit slice*; S_{bot}, the 3-punctured sphere, is fixed, while the once-punctured torus S_{top} is allowed to vary over all possibilities, making the slice a representative of the once-punctured torus Teichmüller space. There are countably many ways to pinch S_{bot}, and each way gives rise to a "slice".[6] The space $\mathcal{QF}_1$ is the quasiconformal deformation space of H_1. It is not composed of quasifuchsian groups; rather $\mathcal{QF}_1 \subset \partial\mathcal{QF}$. The modular tessellation provides a model for the boundary $\partial\mathcal{QF}_1$, with the Farey series used to index the cusps.

For Teichmüller theory the most useful slices have been the *Bers slices* that are submanifolds of $\mathcal{QF}$. The closure of a Bers slice is homeomorphic to a disk, whereas the closure of a Maskit slice is homeomorphic to a disk minus a boundary point (Minsky). A Bers slice is determined by fixing the conformal type of the once-punctured torus S_{bot} while allowing S_{top} to vary. McMullen showed that cusps are dense on its boundary. Computing a Bers slice is more difficult, since doing so involves

[4] *Actually the group H_0 first appeared, not in Jorgensen's* Annals *paper (as suggested in the book), which featured a family of degenerate groups with elliptic commutators, but in a somewhat later, related paper with the reviewer.*

[5] *There is only one 3-punctured sphere in the sense that any two are Möbius equivalent.*

[6] *There are also slices based on a singly degenerate S_{bot}.*

The Mathematical Cartoonist

The cartoons in *Indra's Pearls* are the creation of freelance cartoonist Larry Gonick. That they convey mathematical ideas so well is no accident: Gonick was once on track to become a mathematician, studying mathematics as an undergraduate and graduate student at Harvard University in the 1960s and 1970s. Today he is known for his wacky, brainy cartoons about science, mathematics, and history.

Gonick began his cartooning career with *Blood from a Stone: A Cartoon Guide to Tax Reform* (New York Public Interest Research Group, 1972), written with Steve Atlas. "It was the dullest subject my coauthor could think of, so it needed cartoons," Gonick remarked. He went on to write political and historical cartoons for Boston newspapers. He moved to San Francisco in 1977 and fell in with the world of "underground comics", a genre exemplified by the work of R. Crumb. Starting in the early 1980s, Gonick coauthored a series of cartoon guides to such subjects as genetics, physics, and the computer. The bestseller of the series is *The Cartoon Guide to Statistics* (HarperCollins, 1994) by Gonick and Woollcott Smith, which has been widely used as a classroom supplement and as a training tool in industry. Gonick has also written a series of cartoon histories; the most recent is the third volume of *The Cartoon History of the Universe* (W. W. Norton), which appeared in fall 2002.

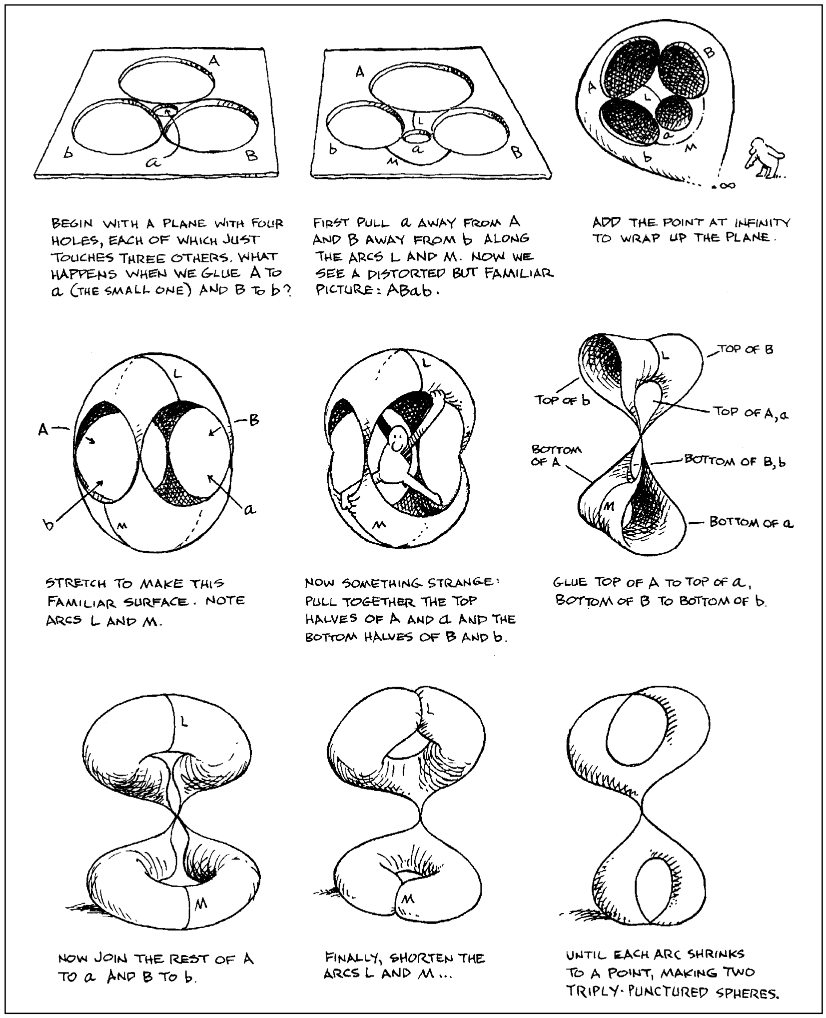

Figure 3. Dr. Stickler is showing how a pair of thrice-punctured spheres is obtained from the exterior of four Schottky "kissing circles" (Fig. 7.7 in the book).

While in college, Gonick got to know David Mumford, one of the authors of *Indra's Pearls*. The two reconnected when Gonick spent 1994–95 as a Knight Science Journalism Fellow at the Massachusetts Institute of Technology, and they collaborated on software designed to visualize four-dimensional space. A few years later, Mumford told Gonick that the 20-year-long project of writing *Indra's Pearls* was finishing up and sent him a copy of the manuscript. "I looked at it and I thought, 'No way,'" Gonick recalled. "If it took them twenty years to do this, it would take them another seven years to finish it. It turned out [Mumford] was right, and I was wrong." Mumford asked Gonick to do some cartoons for the book, and Gonick agreed.

Mumford would send Gonick sketches outlining the ideas the cartoons were supposed to convey. Gonick's use of the figure of "Dr. Stickler" adds a human touch to the drawings and conveys a tactile sense of movement. For example, the first cartoon in the book shows how to make a torus out of a square by identifying the opposite sides of the square. By having Dr. Stickler manipulate the square, with a little pot of glue at hand to stick the sides together, Gonick just about eliminates the possibility that a reader could get confused. The cartoons are not merely accurate; they are perhaps the best way of giving the reader intuition about topological ideas.

The humor and whimsy in the cartoons are not add-ons but derive from the nature of mathematics. During his stint drawing two-page cartoon strips for the science magazine *Discover*, Gonick created several strips based on mathematical ideas, such as transparent proofs, factoring, and DNA computation. The editors did not give him a hard time about strips on deep mathematics. "I was usually able to convince them that the more recherché the subject, the better the strip," he recalled. "Sometimes they would come in with ideas for the strip that were based on the kookiness of an experiment, and I'd say, 'No, there's not enough humor in the experiment being kooky.' There has to be some deep principle at work. That's where the humor comes from."

—Allyn Jackson

numerically solving Schwarzian equations and computing monodromy; nevertheless, this has recently been done by the team of Komori, Sugawa, Wada, and Yamashita.

Return to the group H_1 obtained by pinching $S_{bot}(G)$ and the Maskit slice $\mathcal{QF}_1$ that it determines. We can pinch once more, heading off to a cusp on $\partial\mathcal{QF}_1$: Choose any simple loop $\gamma_t \subset S_{top}(H_1)$ that is not in any parabolic conjugacy class; in particular, γ_t and γ_b do not determine the same conjugacy class within H_1. Pinch γ_t. We end up with a group $H_2 \in \partial\mathcal{QF} \cap \partial\mathcal{QF}_1$ such that not only Ω_{bot} but now also Ω_{top} has become a countable union of round disks. Topologically, the interior $\mathbb{H}^3/H_2$ remains homeomorphic to $S \times (0,1)$, with S a once-punctured torus, while $\partial\mathcal{M}(H_2)$ is the union of two 3-punctured spheres (see Figure 3). The group H_2 is rigid; it cannot be deformed (modulo Möbius equivalence). The limit set $\Lambda(H_2)$ is the union of circles and limits of circles. Particularly significant in understanding the limit set are the parabolic fixed points; these are the points of tangency of the circles.

The authors introduce groups of the type H_2 by means of the *Apollonian gasket,* represented in Figure 7.3 as a beautiful "glowing gasket." It is constructed by means of a symmetric arrangement of the four "kissing circles", which determines three circular triangles. The cartoon Figure 3 well displays the process of pinching leading to a double cusp group.

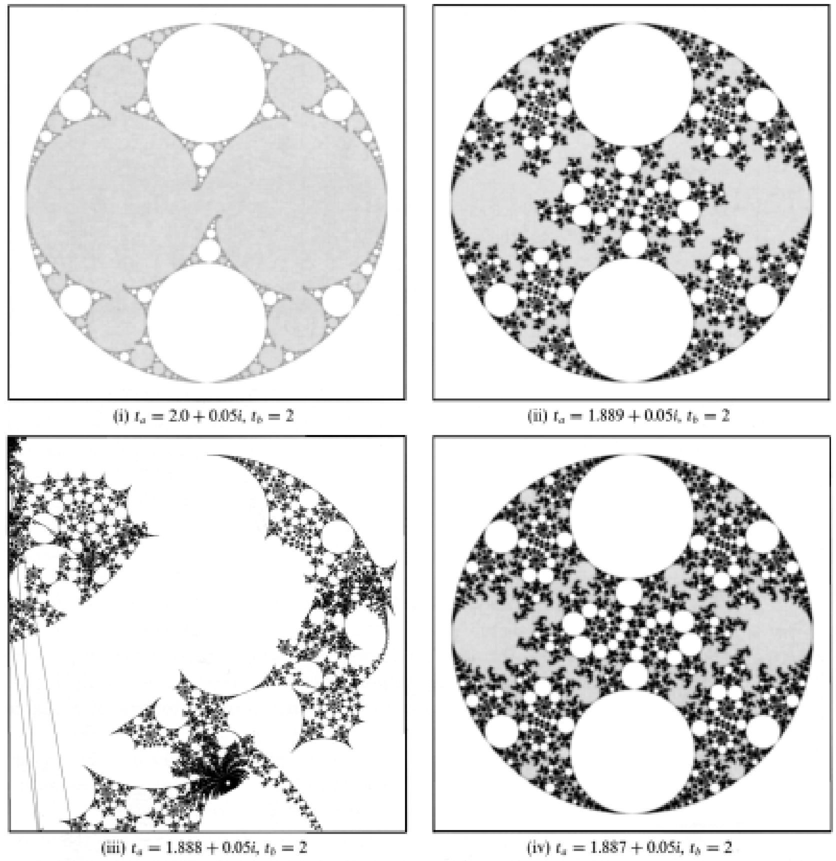

Figure 4. Exploring a Maskit slice $\mathcal{QF}_1$ near a double cusp group of type H_2 on $\partial\mathcal{QF}_1$. The group whose limit set is (i) is close to H_2 while in (iii) the group is just outside $\mathcal{QF}_1$ and is not discrete. The parameters t_a, b_b are traces of generators (Fig. 8.15 in the book).

The first seven chapters (two-thirds of the book) consist of introductory material and detailed discussion of a number of typical examples. Included is a very clear presentation of the modular tessellation and Farey series with related continued fraction expansion. Later in the book it is explained how the rational numbers correspond to slopes of simple curves on a torus and words in the generators.

Chapter 8 begins by introducing a number of parameter systems, involving the trace of two felicitously normalized generators, for $\mathcal{QF}$. These are used to roam about the space, with particular attention paid to the effect on the pictures of changing the generator traces. We find out how to "tighten the spirals" (Figure 8.13) and "raise a crop of spirals" (Figure 8.23). The subtle matter of visualizing "thin necks" is addressed (Figure 8.17). We learn how to estimate the Hausdorff dimension of the limit set. And we are led out to "single cusps" of the type H_1 and "double cusps" of the type H_2 on the boundary.

Chapter 9 is devoted to exploring the boundary of a Maskit slice. Guided by the Farey series indexing of the cusps (now of type H_2), the boundary is traced out. The Schottky heritage of these cusps is still evident in the form of "smushed" Schottky circles (Figure 9.5). In describing the cusps the authors use the term "accidental parabolic", which has come into common use for a "new" parabolic that arises in a boundary group. This is unfortunate terminology, because building and finding such groups is anything but "accidental".

Chapter 10 fills out our knowledge of a Maskit slice boundary. We have a look at the boundary itself and notice its self-similarity (Figure 10.3). We are shown how to approach an irrational (non-cusp) boundary point by a sequence of double cusp groups. We are shown two examples of *singly degenerate groups* (Figures 10.4, 10.6), with indication of how the degeneration of the regular set has occurred: In these examples the top punctured torus boundary component of the 3-manifold has vanished completely, leaving the triply punctured sphere on the bottom. The limit set now has Hausdorff dimension two (Bishop-Jones) but still no interior. In fact, the double cusps of type H_2 are dense on $\partial\mathcal{QF}_1$ (Canary-Culler-Hersonsky-Shalen), and all the other boundary groups are singly degenerate.

We are taken back to quasifuchsian space $\mathcal{QF}$ for an approximation to the *doubly degenerate group* H_0. It is found as the limit of a cleverly symmetrized sequence of double cusp groups on $\partial\mathcal{QF}$. Here both once-punctured tori components of the 3-manifold have vanished. The limit set has become $\mathbb{S}^2$ itself; an approximation appears in Figure 10.1. One finds the original structure buried in the limit set as the lift to $\mathbb{H}^3$ of each fiber $S \times \{x\}$ determines a space-filling curve on $\mathbb{S}^2$ (Cannon-Thurston). By the use of the group, the authors show how to partition $\mathbb{S}^2$ in a way to reflect its Schottky heritage.

Lest the reader come away with the idea that he has understood everything, the final examples should dispel any such thought. There is a striking example of a geometric limit properly containing the algebraic limit at a double cusp (Figure 10.16). Jorgensen's theory could be repeated by requiring the commutator $[A, B]$ to be elliptic of finite order rather than parabolic. Figure 11.1 is an example of a cusp group of this type. For this example the Schottky circles intersect at $45°$ (Figure 11.2). To cope with the group theoretical complications in developing an algorithm to compute the limit set of such a group, the authors introduce the theory of automatic groups. In this respect, they use a program of Derek Holt that has implemented the theory.

It makes a huge difference that the pictures are of the highest resolution and quality in that we have a clear view for long distances into the fine detail of the limit sets. The many diagrams are drawn with great care, both as to visual impact and to their content. The colorations are not only artful but are used to bring out special mathematical properties.

The authors intend that the book be read and understood by a broad audience, by everyone who is comfortable with high school algebra. A broad audience could certainly understand the earlier chapters. The complexity builds up slowly; techniques and ideas are explained from first principles as they are introduced. But toward the end, less mathematically experienced readers might need help. In the academic world the book could be used as the basis of a college course on "chaos and fractals", or read as an introduction to group actions, or read just for enjoyment by graduate students and, not least, by professors. At the end of each chapter there is a well-chosen set of problems called "projects" that provides the reader with hands-on experience with the theory.

An important aspect of the book is its inclusion of pseudocode which in principle will allow a savvy programmer, or "hacker", to explore for himself the space of these groups. But even a reader who is not interested in computing will find the pseudocode essential in understanding how the pictures are made. The computational aspect is a natural vehicle to draw those with hacker ambition into the mathematical content. Conversely, a mathematical reader who has become intrigued while reading about the subject may be enticed to make his own computer explorations.[7]

Indra's Pearls is not written in the theorem/proof style of a mathematics textbook. Rather, it is a flowing narrative, leavened with wit, whimsy, and lively cartoons by Larry Gonick. A delightful quotation highlights a theme for each chapter. For example, Chapter 3 begins with a quote from Lewis Carroll:

> First accumulate a mass of Facts: and *then* construct a Theory.
>
> *That*, I believe is the true Scientific Method.
>
> I sat up, rubbed my eyes, and began to accumulate Facts.

The chosen style exposes the authors' own love of, and long experience with, the subject, which they have made a great effort to communicate.

The heaven of the great Buddhist god Indra is said to be an infinite web strung with pearls. In each pearl all the others are reflected, in each reflection the infinite number of pearls is seen again. Indra's universe is seen in each pearl and seen over and over again on smaller and smaller scales; the metaphor of the title anticipates the geometry. The three authors, with the support of Cambridge University Press, have produced a book that is as handsome in physical appearance as its content is stimulating and accessible. The book is an exemplar of its genre and a singular contribution to the contemporary mathematics literature.

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[7] *Masaaki Wada has built a publicly available Mac program, OPTi, which allows the user to explore $\mathcal{QF}$, as parameterized by Jorgensen's "complex probabilities". OPTi in particular indicates the Ford polyhedron corresponding to each point of $\mathcal{QF}$. For information on OPTi, see* `http://vivaldi.ics.nara-wu.ac.jp/~wada/OPTi/index.html`.