Mathematical Reflections *and* Mathematical Vistas

Reviewed by T. W. Körner

Mathematical Reflections
Peter Hilton, Derek Holton, and Jean Pedersen
Springer-Verlag, New York, 1997
351 pp., ISBN 0-387-94770-1, $44.95

Mathematical Vistas
Peter Hilton, Derek Holton, and Jean Pedersen
Springer-Verlag, New York, 2002
335 pp., ISBN 0-387-95064-8, $59.00

Mathematical Reflections and *Mathematical Vistas* are two books of mathematical essays intended for able "secondary students of mathematics, undergraduate students of mathematics or adults seeking to extend their mathematical competence." The authors have tried to make the two books independent, but anyone who likes one will like the other, and it makes sense to read them in the order in which they were written (that is, first *Reflections* and then *Vistas*). A list of chapters is given in the sidebar.

The readers of this review will not, in the main, form part of the specified audience, but they do choose books to stock the libraries used by that audience. The important part of my review is thus contained in the next sentence. These are excellent books which should be in every university, college, or school library used by actual or potential students of mathematics. When you have ensured that your institution's library has obtained the two books, you should look at them and decide for yourself whether (as may well be the case) you might learn something by reading them. Let me add that Springer has done an excellent job of presentation and that though one would wish the prices of these books lower, they are not unreasonable.

There are many difficulties involved in the project the authors have undertaken, and most of the rest of the review is an extended reflection on them. This may give the review a slightly somber tone, so let me emphasise that I believe that the authors have done a splendid job of overcoming the difficulties. If I could write books like these, I would.

Now let us look at some difficulties.

Reviewers: Who should review books of this type? (From now on I shall call them semipopular books.) Obviously not students. They have no standard of comparison and no way of telling if the authors achieve their intended goal. Professors trying to be the eighteen-year-olds they once were? So far as I can remember my eighteen-year-old self, I think he would have enjoyed these books but failed to understand much of what was said. Much of what the eighteen-year-old did understand would be later forgotten, but some arguments would remain, like beacons illuminating his mathematical life. In other words, these books would be a splendid experience. However, my middle-aged self cannot tell which bits would be the beacons—perhaps the idea of a contraction mapping from the chapters on paper-folding or perhaps the Pólya enumeration theorem. The only remaining reviewers are professors. But professors read like jackdaws, picking out the glittering bits. (I have never been good at explaining why

T. W. Körner is a professor in the Department of Mathematics and Mathematical Sciences, Trinity Hall, University of Cambridge, England. His email address is `T.W.Korner@dpmms.cam.ac.uk`.

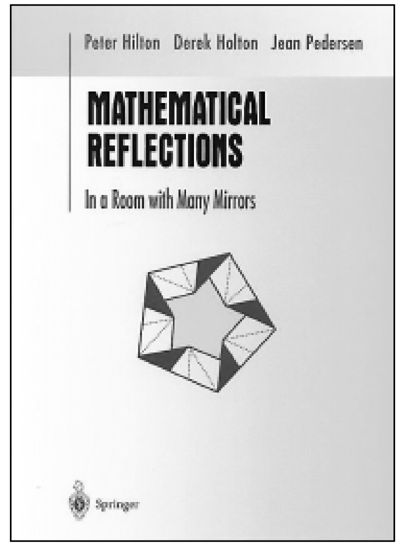
Euler's totient function is multiplicative. After reading the appropriate part of Chapter 2 of *Reflections*, I now know how to do it.) They also read what they think is written and not what is written. It is impossible to find completely appropriate reviewers for semipopular mathematics texts.

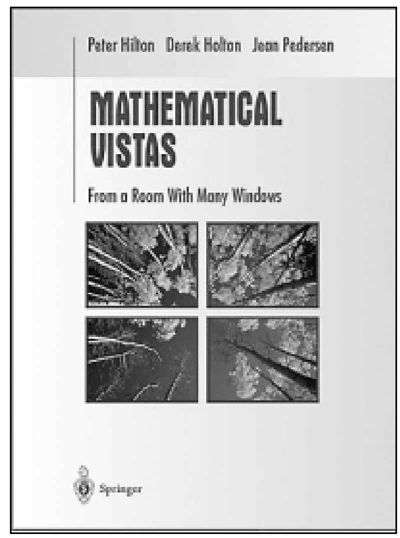
Where to start: The lecturer to a second-year or third-year university class knows what the students are supposed to know and what notation can be safely used. The writer of a semipopular text enjoys no such advantage. The authors of this text have used two classical techniques to get round the problem. The first is to make the various essays independent, so that a student who does not get on with one topic can switch to another. The second, which they employ extremely gracefully, is to start each chapter at a fairly low level and gradually increase the difficulty. Of course, however smoothly the increase in difficulty is handled, the student must eventually reach the limit of his or her capabilities, and this means that few will reach the end of any chapter with full understanding. My eighteen-year-old self tells me that this does not matter, and my middle-aged self agrees.

The problem of where to start can be mitigated, but there is no way of resolving it entirely. In the first chapter of *Reflections* the authors need a polar equation of the form $r = \theta$. If θ is measured in degrees, the picture is peculiar. If θ is measured in degrees and the equation replaced by $r = \theta/100$, we have introduced an artificial constant. Instead, the authors choose to be consistent with standard practice and measure θ in radians. They give a brief explanation of radian measure, but, whilst their explanation would function well as a reminder, I doubt whether it would work as an introduction. Elsewhere they assume that the student can easily use the sum symbol $\sum_{i=1}^{n}$, and more generally they require a quite high level of fluency in algebraic manipulation. It seems to me that overall their choices are wise and consistent but limit the readership to "university-ready" school children and above.

Where to finish: The lecturer to a first-year or second-year class knows what the students are supposed to know at the end of the course. The structure that the lecturer builds is primarily intended as a foundation for some well-defined future course. The writer of a semipopular text has no such well-defined goal. The authors of these two books have tried to make each chapter complete in itself and have, I think, succeeded, but inevitably tensions remain.

Some of the chapters introduce topics that form part of every mathematician's education (polar coordinates, modular arithmetic, countability, groups, and graphs). I would be delighted if my own students had read these chapters before they arrived at university. Other chapters, although enjoyable in themselves, would be less useful. As the most extreme example, there are two chapters on mathematical constructions using paper-folding. These have many virtues. They bring the student into direct contact with new research, for the results were obtained by two of the authors. The ideas could form the basis of delightful hands-on classes for school children of all ages. However, the mathematics involved is real mathematics and like all real mathematics takes hard work to really understand. It would be a valuable experience for my students to master this material, but it would be both a valuable and *useful* experience for my students to master div, curl, and grad before they arrived at university. If the authors were to reply that prospective school teachers need exposure to the idea of mathematical research much more than a lot of tedious manipulation of vector calculus identities, I would not disagree. Different audiences have different needs, and a semipopular book cannot confine itself to one audience.





The professor over your shoulder: There is a romantic opinion among American mathematics students and many of their teachers that authors of mathematics textbooks should think only of their readers and never of their peers. The gigantic, now-with-2,000-more-routine-exercises-and-examples-from-business-studies, available-in-early-transcendental-and-late-transcendental, four-coloured, answers-to-odd-exercises, all-American calculus text stands as a monument to this ideal of consumer power.

I do not find it hard to accept that Beethoven cared more for the opinion of Haydn than for that of his paying audiences. I find it easy to imagine Shakespeare laying down his pen at the end of Hamlet's "To be or not to be" and saying, "That will show them how to write a soliloquy" or Silvanus P. Thompson muttering from time to time "And this is the way to teach calculus." In the same way, I could hear the three authors gently chorusing, "This is how you give a proper lecture on fractals, and this is how you give a talk on the proof of Fermat's last theorem." However, the "lessons for teachers" are left implicit and nowhere interfere

Below are the chapter titles for the two books under review.

Mathematical Reflections
Going Down the Drain
A Far Nicer Arithmetic
Fibonacci and Lucas Numbers
Paper-Folding and Number Theory
Quilts and Other Real-World Decorative Geometry
Pascal, Euler, Triangles, Windmills,…
Hair and Beyond
An Introduction to the Mathematics of Fractal Geometry
Some of Our Own Reflections

Mathematical Vistas
Paradoxes in Mathematics
Not the Last of Fermat
Fibonacci and Lucas Numbers: Their Connections and Divisibility Properties
Paper-Folding
Polyhedra-Building and Number Theory
Are Four Colors Enough?
From Binomial to Trinomial Coefficients and Beyond
Catalan Numbers
Symmetry
Parties

with the authors' duties towards their primary audience.

The only point where they address the professor directly is in the chapter entitled "Some of Our Own Reflections". My preferences go elsewhere, but many people will find it one of the most interesting chapters in the two volumes. The first half is addressed to the student and asks, How should mathematics be done? The advice is unobjectionable (particularly if one adds the words "however, there are many exceptions" to each recommendation) but does not produce the impact that Pólya achieves by a happy marriage of precept and example.

The second half states some "Principles of mathematical pedagogy". The final recommendation "Never cut short an explanation or exposition in order to complete an unrealistically inflated syllabus" should be inscribed in letters of gold above the blackboard of every mathematical classroom. Indeed, I would save money on gold leaf by leaving out the two penultimate words. However, I would hesitate to send prospective university teachers out into the real world armed only with the rest of the authors' somewhat optimistic advice. (I would recommend instead the more worldly *How to Teach Mathematics* by Steven Krantz.)

The authors state their "basic principle of mathematical instruction" as follows: "Mathematics should be taught so that students have a chance of comprehending how and why mathematics is done by those who do it successfully." Obviously there are problems in applying this advice when teaching the class described in Berlinski's *Tour of the Calculus* (not the class from hell but the class from *The Education of Hyman Kaplan*) or, at the other extreme, the young wolves of the Independent University of Moscow, but there are also problems for more usual classes. Suppose that we ask mathematics students what they want from their mathematics class. Human beings being what they are, some students will have contradictory or deeply unrealistic expectations. Others will have clear and achievable goals. "I need an A to go to medical school." "I need to master calculus to do physics." "University represents the only freedom I expect to have in my life. I need a C to stay on." These goals are not ignoble, but they do not run in parallel with the goals implied by the "basic principle".

To this objection the authors could well reply that the clash between what people want and what they need occurs everywhere in human life and though easy to state is impossible to resolve. They might also remark that "Drive as though everyone else is drunk" is a good precept even though based on a false premise. Even those who disagree with their theories must admit the excellence of the authors' practice.

Exercises: There is a *New Yorker* cartoon in which a mother tells her child, "It's broccoli, dear," and the child replies, "I say it's spinach, and I say the hell with it." The authors use a device which they call a "break", but which I would call an exercise. As far as I could judge, these breaks are well chosen and would be useful if the chapters were used as part of a course or as the base for lectures with audience involvement.

I am less clear if exercises are useful for the individual reader. I suspect that my eighteen-year-old self would have skipped them or peeked at the answers. These are books which demand to be read with paper and pencil at the ready, and any serious reader will find plenty of work to do in understanding the contents without looking for further challenges. Remember that books like these are meant to be read for pleasure and not as a kind of weightlifting exercise. More generally, I cannot help reflecting that whilst professors are strongly in favour of lots of exercises for their students, the books they write for each other rarely contain exercises.

Ideology: If it is not already evident from my earlier comments, let me state my opinion that a book with a strongly held point of view is likely to be better than one that tries to satisfy everyone. The authors believe that mathematics is to be valued for its beauty rather than its utility, and although the introduction to *Vistas* claims that the examples are drawn from both pure and applied mathematics, this British critic would classify almost all the material as pure. The only point where I thought that the authors' point of view weakened their exposition was in the chapter on "Paradoxes in Mathematics", more particularly in Section 1.5 (on Simpson's paradox in statistics) and Section 1.6 (can "physical" models use "unphysical" elements). These were the only places in the two books where I felt like wresting the pen from the authors and saying, "You have a good point, but you are not making it properly." In both cases I felt that the presentation was too abstract to convince students that the arguments had any relevance outside the classroom.

Reaching the audience: When I discussed the authors' views on pedagogy, I used a crude rhetorical device to imply that because most students failed to share the ideals of their "basic principle of mathematical instruction", no students shared those ideals. Now let me apologise and admit that there are many people in the world who wish to know "how and why mathematics is done by those who do it successfully."

Such people are a small minority in most mathematics classes, but there are many mathematics classes in the world. Such people may also be found, as a still smaller proportion but now of a much larger number, among nonmathematicians. Many among these interested outsiders recall the reported lament of Shostakovitch, "I would want my music only played by amateurs…if only amateurs could play." However, many others are physicists, computer scientists, secondary teachers, engineers, and so on who combine a simple curiosity about mathematics with the level of technical ability required by these books.

The ideal audience envisaged by the authors' "basic principle" exists and is quite large. However, it is also dispersed, and this creates serious problems. Let the reader, if he or she belongs to a university, go down to the university bookstore and look through all the mathematics books available (semipopular books, course texts, and specialist texts combined). Unless he or she is very lucky, the results will be rather sobering. (A walk through a large generalist bookstore is also rather sobering, but we bear less responsibility for the mathematical culture of the general public than we do for the mathematical culture of our own students.)

With our bookshop stroll in mind, it should be clear why we have a duty to be sure that the mathematics library of our institution buys books like these. Too often we think only of recommending books that students or staff *ought* to read and forget those that they might *like* to read.

What about that part of the authors' ideal audience with no university connections? Here there has been a major change for the better. Whether the great wheel of fortune of U.S. capitalism finally leaves Jeffrey Bezos in a palace or a (metaphorical) hovel, there is no doubt that he deserves a double handful of honorary degrees from the mathematical community. The few of us who live near great mathematical bookshops may treasure the luxury of holding potential purchases in our hands, but for the rest the electronic bookshop is a major advance in civilisation. Thanks to the electronic bookshop and related innovations, books like these can remain available for many years. (*Reflections* is already in its second printing.)

There are many things to depress us in mathematical education: students who want routine teaching for routine examinations and staff only too willing to oblige, well-meaning governments who seek changes in twelve months that require twenty years of goodwill, armies of advisers who know everything about mathematics teaching except the mathematics.… The list is long. The authors of these two books have decided that it is better to light a candle than to curse the darkness, and their two candles give a very fine light.