The Zen of Magic Squares, Circles, and Stars: An Exhibition of Surprising Structures across Dimensions

Reviewed by Andrew Bremner

The Zen of Magic Squares, Circles, and Stars
Clifford Pickover
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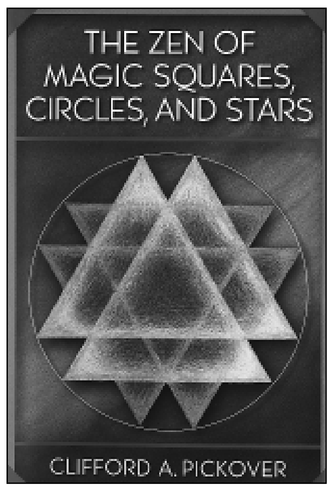
Benjamin Franklin, the youngest son and fifteenth in a family of seventeen children, is renowned for his statesmanship and for his work on electricity. But he has a lesser-known claim to fame, the ability "to fill the cells of any magic square, of reasonable size, with a series of numbers as fast as I can write them, disposed in such a manner, as that the sums of every row, horizontal or perpendicular, or diagonal, should be equal." Such was his facility that one of the outstanding known examples of magic square was constructed by him in a single evening, and he was proud of its intrinsic beauty: "You will readily allow this square of 16 to be *the most magically magical* of any magic square *ever* made by any magician."

Magic squares are ancient and common to several civilizations. They were sometimes used as amulets and viewed as possessing talismanic powers. The first references appear to be Chinese, with the familiar 3×3 square

$$\begin{pmatrix} 4 & 9 & 2 \\ 3 & 5 & 7 \\ 8 & 1 & 6 \end{pmatrix}$$



known as the lo shu. Magic squares can be traced in early Arabic and Indian cultures, appearing in the former towards the end of the first millennium. They began to appear in European writings apparently around the fifteenth century and by the eighteenth century had attracted the study of several well-known mathematicians, including Faulhaber (the "Arithmetician of Ulm" and original discoverer of the Bernoulli numbers) and Euler. There followed a surge of writings. David Singmaster has one of the world's best collections of recreational mathematics and puzzle books (the others are the collections of Will Shortz, editor of the *New York Times* crossword puzzle, and the Strens-Guy Collection at the University of Calgary). Singmaster's source bibliography [5] contains eighteen pages of exhaustive references to magic squares and figures, and the explosion of texts in later centuries is reflected by the statement that "17th–20th century material has generally been omitted." In the current volume Pickover starts out with his own brief history of magic squares, which is perhaps not quite so scholarly in its reference and which aims for popular appeal.

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So we have a delightful description of the eighth-century Islamic alchemist "Geber", who in his attempts to transmute base metal into gold evidently used elixirs based on numbering various substances (including gazelle urine) according to the numbers in his magic square; however, there is no attribution. One passage on page 11, extracted verbatim from what must be a rather dubious source, is startling: "We find it [the lo shu] venerated by civilizations of almost every period and continent. The Mayan Indians of southern Mexico and Central America were fascinated by it, and today it is used by the Hausa people of northwestern Nigeria and southern Niger as a calculating device with mystical associations. The square was respected by the ancient Babylonians and was used as a cosmic symbol in prehistoric cave drawings in northern France." This latter assertion in particular seems quite preposterous. There *are* no Paleolithic cave drawings in the north of France. More fundamentally, the specific existence of a symbol system at this early date is questionable and is the subject of a large and complex area in Paleolithic art research.

It is not clear when the magic square first appeared in English writings. Consulting the *Oxford English Dictionary*, we find the first recorded mention of "magic square" to be an entry in a technical lexicon of 1704. Around this time the adjective "magic" was shifting in meaning towards that of "producing wonderful appearances or results, like those commonly attributed to sorcery" as opposed to the older sense of directly describing the art of the occult, with its intervening spirits or controlling principles of nature. According to Singmaster, one of the first European manuscripts to deal with magic squares is a fifteenth-century Latin manuscript in Cracow (which incidentally gives the famous 4×4 square that occurs in Dürer's print *Melencolia*). The sixteenth century provides a couple of additional references in Latin but also some in German, and Frénicle de Bessy made major contributions in "Des quarrez ou Tables Magiques" of the late seventeenth century. Presumably the English phrase entered the language as a direct translation from either French or German, with "magisches Quadrat" harking back to when magic squares were genuinely associated with the arts of divination and alchemy. Goethe in *Faust* [3] constructs a mysterious magic square that he calls the Hexen-Einmaleins, a direct link to a sorcerous source ("Du mußt versteh'n! / Aus eins mach zehn, / Und zwei laß geh'n, / Und drei mach gleich, / …").[1]

Pickover has written a splendid recreational book, though it contains very little actual mathematics. The author does say in his introduction that "This book is not for mathematicians looking for formal mathematical explanations," and indeed the number of proofs given is minimal. He does, for example, include demonstrations of the propositions that there do not exist perfect magic cubes of orders 3 and 4, but in general the thrust of the book is to provide numerical examples of ever-increasing complexity in a context of entertaining prose for the lay reader. An N*-th order* magic square is an $N \times N$ square containing the integers $1, \ldots, N^2$, with the property that all row, column, and principal diagonal sums are equal. Up to symmetry there are 1, 0, 1, 880, 275305224 magic squares of orders 1 to 5 respectively. It has been estimated (see [4]) that the number of magic squares of order 6 is 1.77×10^{19}. You can find all 880 squares of order 4 in Frénicle de Bessy (posthumous work of 1693) or, more accessibly, categorized in Berlekamp, Conway, and Guy [1]. Pickover's interest, however, lies outside the simple N-th order squares, and we are treated to an amazing menagerie of oftentimes fantastic formations.

It is possible in a brief review to convey only a fraction of the sheer exuberance and variety of squares on display. You will find squares of primes, of consecutive primes, and of consecutive composite integers. There are simultaneous addition and multiplication squares, where multiplication along rows, columns, and diagonals also produces a constant product. There are squares that remain magic when each element is replaced by its square (*bimagic*), and similarly squares that remain magic when elements are replaced by the square or cube (*trimagic*). (In fact, there are recent analogous results (Boyer & Viricel [2]) showing the existence of *quadrimagic* and *quinquemagic* squares, the latter example being of order 1024.) A 10-digit integer is *pandigital* if it contains each of the digits 0, …, 9; an example is given of a 3×3 square with pandigital entries and pandigital sum. There is a 4×4 square (whose entries are formed from 1's and 8's) with the property that it remains magic when viewed upside down or in a mirror. And these are only the two-dimensional specimens. The range of examples for cubes, tesseracts, circles, spheres, stars, and bizarre geometrical configurations in general (including a magic spider) is extreme. There is even a magic hypercube in five dimensions.

The book in consequence has an appealing visual nature: open it at random, and the eye will be dazzled. The author indeed finds a fruitful source of geometrical patterns in Buddhist mandalas and other meditative images. Using the term *satori* from Zen Buddhism, which the dictionary defines as "a sudden indescribable and uncommunicable inner experience of enlightenment," the author writes that "*Arithmetic satori* is the psychological result and aim of the practice of magic square meditation" and that this practice "induces an awareness, an

[1] *"Now you must ken! From one make ten, And two let free, Make even three …."*

experience of joy emanating from a mind that has transcended its earthly existence." He states in "Some Final Thoughts", page 373, that "while studying and doing research on magic squares, sitting in my home office or in a library or while gazing at the computer screen, I often get a flicker of happiness, or dare I say, 'transcendence' or 'wonder', that seems to bring the magic square to life…for a few seconds, I felt touched and mystically elevated." Perhaps Pickover is just experiencing the primal pleasure in the process of pattern recognition or feeling a thrill similar to that of the professional mathematician as the components of an abstract proof fall into place. Clearly, arithmetic satori will not come to all readers, and the reviewer, for example, on fixedly studying the "super overlapping fifteenth-order magic square" has managed only a state of hypnagogic lassitude. Enlightenment is something that will take a great deal of time and effort! But this is facetiousness on my part, and the author's research and meditation does provide a great treasury of entertaining material.

Some of the figures truly pop the eyes, though many have a certain *arbitrariness* that is vaguely unsettling. The intrinsic underlying mathematical structure of the square, cube, and tesseract seems lacking from the "Circles of Prometheus", the "Cirri of Euripides", and the magic spider. Nonetheless, it is the examples that render the book most valuable and at the same time provide a salutary lesson in what extraordinary insight and talent the mathematical "amateur" can possess. Many of the stunning squares, cubes, and higher-dimensional forms were discovered by John Hendricks, a former employee of the Canadian Meteorological Service who retired in 1984. Several other examples were constructed by prison inmates (notably, a 7×7 square of primes, remaining magic when the rightmost digit of each entry is deleted). It is intriguing to ponder the reaction of our nation's sheriffs to the author's penological rumination: "One wonders what effect there would be on magic square research, mathematics, and society if prisoners were rewarded for any novel magic squares they created."

Occasionally one glimpses intriguing aspects of underlying mathematics, for instance in the following remark. Identify a magic square with its corresponding matrix, so that a multiplication is defined between magic squares of the same dimension. Then the cube of a 3×3 magic square is also a magic square! (This highly nonobvious observation is due to Frank E. Hruska, professor of chemistry at the University of Manitoba.) For instance, if A is equal to the lo shu square of sum 15, then

$$A^3 = \begin{pmatrix} 1149 & 1029 & 1197 \\ 1173 & 1125 & 1077 \\ 1053 & 1221 & 1101 \end{pmatrix},$$

which is a magic square of constant sum $3375 = 15^3$. The author remarks that "raising any third-order magic square to any odd power seems to yield a magic square. (We used a computer to test this up to the fifteenth power.)" A proof is furnished for the third power, using a parametrization of all 3×3 magic squares (curiously, the author states that such a square may be parametrized in one of two forms; he displays the forms apparently without realising that they are directly equivalent). However, this fascinating thread is not developed further.

A dutiful student of linear algebra recognises that if M is a matrix representing a magic square, then necessarily both M and M^t have the vector $(1, 1, \dots, 1)$ as eigenvector, with corresponding eigenvalue the constant magic sum (which equals the trace of M). So it follows immediately that if M_1 and M_2 are magic squares with constant sums m_1 and m_2, then $(1, 1, \dots, 1)$ is an eigenvector of both M_1M_2 and its transpose, with eigenvalue m_1m_2. Equivalently, M_1M_2 is semimagic (has row and column sums all equal, namely, equal to m_1m_2). What happens to the two diagonal sums is less obvious, but for 3×3 magic squares we can resort to the general parametrization and observe by direct computation that the product of three such magic squares is always magic, with constant sum equal to the product of the three original sums. An induction argument now proves that the k-th power (k odd) of a 3×3 magic square of sum a is magic with constant sum a^k.

Similar analysis for 4×4 squares will fail, for it is no longer true that the cube of such a magic square is still magic. However, imposing further symmetry conditions does result in some findings. There are four principal types of magic squares: Simple, Associated, Nasik, and Semi-Nasik. A Simple square has the minimal property of equal row, column, and diagonal sums. In an Associated N-th order square, any two cells symmetric about the centre of the square have sum $N^2 + 1$. In a Nasik square all the broken diagonals also have the constant sum, and a Semi-Nasik square has a less strong condition on the broken diagonals. By playing with a generic parametrization for the 4×4 case, it can be shown that the product of three Associated squares is Associated and that the product of three Nasik squares is Nasik (and hence the cube of an Associated or Nasik square is respectively Associated or Nasik); however, no such property seems to hold for products of Semi-Nasik squares. There is much to investigate here, and more in higher orders. A bright undergraduate student could fashion an enjoyable research experience from this aspect of the subject in itself.

Several errors in the book can be attributed to poor proofreading, with others perhaps ascribable to hurried writing. For example, the "Yin" 4×4 square on page 278 has an entry 227 misprinted as 277, and 2^3 is printed on page 22 as 89 in the "cube" of the Dürer square (and Dürer is referred to as the *fourteenth*-century painter and printmaker, despite the stress given to the Dürer square containing 1514, the year of its composition). The famous Jaina square is found in the temples at *Khajuraho* rather than Klajuraho, and there is a reference to Cantor's Theorem on the *unaccountability* of the real numbers. In the section "Further Reading" there is recommendation of an article by "H. E. Du" in earlier editions of the *Encyclopaedia Britannica*, but this was the abbreviation used by the famous puzzle master Henry Ernest Dudeney. And the classic reference work *Magic Squares and Cubes* by W. S. Andrews is listed as published in 1917, whereas the first edition appeared in 1908. More alarmingly, a fifth-order Nasik square is given on page 70 with the assertion that it has "many marvellous properties," but the properties given, involving sums of squares of row elements, simply fail to hold. One hopes that transcription of the vast number of numerical diagrams throughout the book has been done with more care. Still, for the most part these are the quibbles of a pedantic curmudgeon, and it would be churlish to deny the enormous appeal of this book. It is an extremely alluring page-turner. For anyone who loves numbers and puzzles, *The Zen of Magic Squares, Circles, and Stars* is, to quote Ian Stewart, "compulsive (and compulsory) reading."

References

[1] E. R. Berlekamp, J. H. Conway, and R. K. Guy, *Winning Ways for Your Mathematical Plays*, Academic Press, 1982.

[2] C. Boyer, Les premiers carrés tétra et pentamagiques, *Pour la Science*, no. 286 (August 2001), 98–102; see also `http://www.multimagie.com/`.

[3] J. W. von Goethe, *Faust*, 1 Teil, Hexenküche.

[4] K. Pinn and C. Wieczerkowski, Number of magic squares from parallel tempering Monte Carlo, *Internat. J. Mod. Phys. C* **9** (1998), 541–6.

[5] D. Singmaster, *Sources in Recreational Mathematics: An Annotated Bibliography*, Sixth Preliminary Edition, School of Computing, Information Systems and Mathematics, South Bank University, London, 1993 (SE1 0AA).