An Introduction to Analysis on Metric Spaces

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Of course the notion of doing analysis in various settings has been around for a long time. For the purposes of this article, "analysis" can be broadly construed, and indeed part of the point is to try to accommodate whatever might arise or be interesting, at least in some way. In particular this might include spaces of functions, norms on them, and linear operators, perhaps in connection with complex analysis, differential equations, or Fourier analysis.

The monographs [2], [10], [11] provide excellent starting points for a number of topics along the lines of "analysis on metric spaces", and the introductory survey [22] and those in [1] can also be very helpful resources.

Some general notions

A basic scenario is that of a measure space $(X, \mathcal{A}, \mu)$, where X is a set, $\mathcal{A}$ is a σ-algebra of subsets of X, and μ is a nonnegative measure on X for which the elements of $\mathcal{A}$ are the measurable sets. It is common to assume that the measure space is σ-finite, which is to say that X can be expressed as a countable union of measurable sets of finite measure, to ensure certain kinds of nice behavior. On such a space one can define integration, L^p spaces, and so on.

One can add more structure in various interesting ways. For instance, one might have a number of σ-algebras on X, all contained in a large σ-algebra on which μ is defined, and this leads to conditional expectation operators, as in probability theory. In another direction, one might have a bijection T from X to itself such that T preserves the σ-algebra of measurable sets and the measure μ, in the sense that $\mu(T(A)) = \mu(A)$ for all measurable sets A contained in X. This type of situation is studied in ergodic theory.

There is also analysis related to continuous functions, limits, compactness, and so forth, as on a topological space. One can do more on a metric space. Recall that saying that $(M, d(x, y))$ is a metric space means that M is a nonempty set; $d(x, y)$ is a function on $M \times M$ taking values in the nonnegative real numbers; $d(x, y) = 0$ if and only if $x = y$; $d(x, y) = d(y, x)$ for all $x, y \in M$; and the triangle inequality holds, i.e.,

$$d(x, z) \leq d(x, y) + d(y, z)$$

for all $x, y, z \in M$. Let us also make the standing assumption that M has at least two elements.

If $(M, d(x, y))$ is a metric space, α is a positive real number, and $f(x)$ is a complex-valued function on M, then we say that f is *Lipschitz of order α* if there is a nonnegative real number L such that

$$|f(x) - f(y)| \leq L\, d(x, y)^\alpha$$

for all $x, y \in M$. The smallest such constant L is denoted $\|f\|_{\mathrm{Lip}_\alpha}$, and it can also be defined by

$$\|f\|_{\mathrm{Lip}_\alpha} = \sup\left\{ \frac{|f(x) - f(y)|}{d(x, y)^\alpha} : x, y \in M,\ x \neq y \right\}.$$

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Dedicated to Guido Weiss and Eli Stein.

This is not quite a norm, but a seminorm, because $\|f\|_{\mathrm{Lip}_\alpha} = 0$ when f is a constant function. The space of all functions on M which are Lipschitz of order α is denoted $\mathrm{Lip}_\alpha(M)$.

For each point p in M, the function $f_p(x) = d(x, p)$ is Lipschitz of order 1, and $\|f_p\|_{\mathrm{Lip}_1} = 1$. One can derive this using the triangle inequality. More generally, one can check that if $0 < \alpha \le 1$, then $f_p(x)^\alpha$ lies in $\mathrm{Lip}_\alpha(M)$ and $\|f_p^\alpha\|_{\mathrm{Lip}_\alpha} = 1$. However, when $\alpha > 1$, it may be that the only functions in $\mathrm{Lip}_\alpha(M)$ are the constant functions. This is the case when M is a Euclidean space $\mathbf{R}^n$, equipped with the standard metric, because the Lipschitz condition of order α implies that the first derivatives of the function are equal to 0 everywhere.

If M is $\mathbf{R}^n$ and $f(x)$ is a continuously differentiable function on $\mathbf{R}^n$, then $f(x)$ is Lipschitz of order 1 if and only if $|\nabla f(x)|$ is bounded, and $\|f\|_{\mathrm{Lip}_1}$ is equal to the supremum of $|\nabla f(x)|$, $x \in \mathbf{R}^n$. This is not difficult to verify using calculus. When $0 < \alpha < 1$, the property of being Lipschitz of order α is a kind of fractional degree of smoothness, which on Euclidean spaces can be considered in connection with fractional differentiation and integration. See [19]. In any case, the Lip_α spaces of functions have the nice features of being easy to define and making sense on any metric space.

The combination of measure theory and topology entails significant structure. One can start with a set X equipped with a topology that makes X a locally compact Hausdorff space, for instance, and use the σ-algebra of Borel sets (the σ-algebra generated by the open subsets of X) as the σ-algebra of measurable sets on X. For a "regular" Borel measure μ on X, one has nice properties such as density of the space of continuous functions with compact support inside the L^p spaces associated to μ on X, $1 \le p < \infty$.

Let $(M, d(x, y))$ be a metric space, and let $\mathcal{B}$ denote the σ-algebra of Borel sets on M, associated to the topology coming from the metric. Suppose also that we have a regular nonnegative Borel measure μ on M. There is a very interesting compatibility condition between μ and the metric on M, which is that μ be positive and finite on all open balls in M and that there be a positive real number C such that

$$\mu(B(x, 2r)) \le C\, \mu(B(x, r))$$

for all x in M and all positive real numbers r. Here $B(x, r)$ denotes the open ball in M with center x and radius r, defined by

$$B(x, r) = \{y \in M : d(x, y) < r\}.$$

In these circumstances μ is said to be a *doubling measure*.

Actually, there is a more basic doubling condition that one can define on the metric space without referring to the measure. This condition asks that there be a positive real number C' such that for every ball B in M of radius r there exist a family $\mathcal{F}$ of balls of radius $r/2$ such that B is contained in the union of the balls in $\mathcal{F}$ and $\mathcal{F}$ has at most C' elements. One can show that if there is a doubling measure on M, then M is doubling as a metric space in this sense. Note that this kind of doubling property is closely related to Gromov–Hausdorff compactness for families of metric spaces, as in [9].

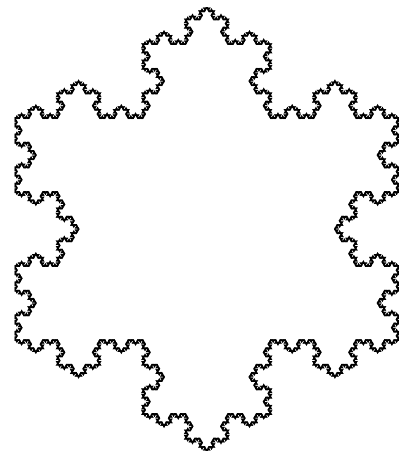

Figure 1. A snowflake curve.

I like to assume also that M is complete, in the sense that every Cauchy sequence in M converges. The doubling condition on M described in the previous paragraph implies that bounded subsets of M are totally bounded, which means that for every positive ϵ a bounded subset of M can be covered by a finite collection of balls of radius ϵ. The assumption that M is complete then implies that closed and bounded subsets of M are compact by a well-known characterization of compactness.

Let us call a complete metric space $(M, d(x, y))$ equipped with a Borel measure μ which is doubling a *space of homogeneous type*, following [3], [4].

Examples

For each positive integer n, the standard Euclidean space $\mathbf{R}^n$ is a basic example of a space of homogeneous type, equipped with its standard Euclidean metric $|x - y|$ and volume measure. On a Riemannian manifold, doubling conditions are connected to lower bounds for Ricci curvature. See [9].

As a more exotic example, consider the space $\mathbf{R}^n \times \mathbf{R}$, equipped with the metric

$$\rho((x, s), (y, t)) = |x - y| + |s - t|^{1/2}. \tag{1}$$

One can check that this is indeed a metric and that ordinary volume measure on $\mathbf{R}^n \times \mathbf{R}$ is doubling with respect to this metric. Just as for the standard metrics on Euclidean spaces, this metric is invariant under translations. However, ordinary dilations do not behave well for this metric, and instead one can use the "parabolic" dilation defined by

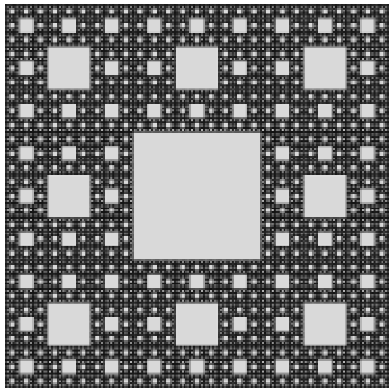

Figure 2. The Sierpiński carpet.

$\delta_r(x, s) = (rx, r^2 s)$ for $r > 0$. This geometry was considered by my colleague Frank Jones in connection with the heat operator

$$\sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} - \frac{\partial}{\partial t}.$$

Roughly speaking, the nonstandard geometry compensates for the fact that there is only one derivative in t, while there are two derivatives in the other directions.

There are numerous familiar examples of self-similar fractal subsets of Euclidean spaces which give rise to spaces of homogeneous type, such as Cantor sets, snowflake curves (Figure 1), and the Sierpiński gasket and carpet (Figure 2). For these one can use the ambient Euclidean metric restricted to the set, although there are often different metrics which are roughly equivalent and which may be adapted to some features of interest. Typically there are also natural measures on the sets which are compatible with the self-similarity and which satisfy the doubling condition.

In particular, the usual middle-thirds Cantor set can be viewed as a geometric model for aspects of probability theory concerning a sequence of coin tosses. One might recall that the Cantor set is "totally disconnected" and hence contains no nontrivial curves. The Sierpiński gasket and carpet are quite interesting for having curves of finite length connecting any specified pair of points, while snowflake curves contain no nontrivial curves of finite length, even if they are themselves curves in the topological sense. A number of aspects of analysis on fractals like the Sierpiński gasket and carpet are treated in [11], [22].

Very interesting examples arise from Heisenberg groups, nilpotent Lie groups more generally, and sub-Riemannian geometry. In the Heisenberg group, the geometry looks like that of (1) on $\mathbf{R}^n \times \mathbf{R}$ infinitesimally at each point, but the axes turn as one moves from point to point. Nonetheless, there are still natural parabolic dilations which respect the geometry on the whole space. For this geometry, there are again curves of finite length connecting any two points. This can be viewed as a special case of sub-Riemannian geometry, in which one starts with a smooth manifold M, a family of subspaces of the tangent spaces of M at arbitrary points, and inner products on these subspaces. One then defines the distance between two points p and q in M to be the infimum of the length of the curves in M connecting p and q, subject to the constraint that the tangent vectors to the curves be contained in the specified subspaces of the tangent spaces to M. These subspaces of the tangent spaces should satisfy suitable "nonintegrability" conditions in order for such curves to exist for all p and q in M. Although the resulting metric is compatible with the usual topology on M, the geometry is quite different.

These examples are closely related to the boundary behavior of functions of several complex variables and subelliptic partial differential operators. See [12], [20] for instance. They also arise in connection with spaces at infinity of rank 1 symmetric spaces of noncompact type. One can define spaces at infinity of complete simply connected Riemannian manifolds of negative curvature more broadly, or even negatively curved metric spaces, and these again lead to examples of spaces of homogeneous type. Compare with [5], [8], [13].

Another fundamental class of examples comes from graphs. Suppose that we have a graph consisting of a set V of vertices, with at least two elements, and a set E of edges. One can think of the edges as being represented by unordered pairs of distinct vertices. We shall assume that the graph is connected, which means that every pair of points can be connected by a finite chain of adjacent vertices. The length of such a chain is defined to be the number of vertices in the chain minus 1, which is the same as the number of edges traversed. This leads to a distance function on the set V of vertices; namely, the distance between two vertices v and w is equal to the length of the smallest chain of adjacent vertices connecting v and w. Thus V becomes a metric space in this way, and we also have a natural measure on V, namely, counting measure, which assigns to a subset A of V the number of elements of A. When this measure is a doubling measure, one gets a space of homogeneous type. For instance, one can take V to be $\mathbf{Z}^n$, with an edge between $x, y \in \mathbf{Z}^n$ exactly when $x - y$ has all components equal to 0 except for one, which is equal to ± 1. Concerning analysis on graphs and related matters, see [14] and the article by Coulhon in [1]. Some further adventures with analysis and combinatorial geometry can be found in [15], [16], [24]. One might wish to look at metric spaces in terms of "nonstandard graphs", in the sense of nonstandard analysis, and this is discussed in [23].

It may be that the graph is finite, and one is interested in quantitative bounds. Some basic examples are given by the natural approximations of fractals like the Sierpiński gasket and carpet by graphs, representing the first n stages of their construction for each positive integer n.

Approximations to the Identity

Fix a positive integer n, and let us review some aspects of analysis on $\mathbf{R}^n$. For each positive real number τ, define a linear operator H_τ on functions on $\mathbf{R}^n$ by

$$(2)\quad H_\tau(f)(x) = \int_{\mathbf{R}^n} (4\pi\tau)^{-n/2} \exp(-|x-y|^2/4\tau) f(y)\, dy.$$

To be more precise, this makes sense if the function f does not grow too fast, e.g., if $f(x)(1+|x|^2)^{-k}$ is integrable on $\mathbf{R}^n$ for some positive integer k.

Well-known computations from calculus imply that H_τ applied to the constant function equal to 1 gives 1 back again. It is easy to see that this is true for all $\tau > 0$ as soon as one checks it for any particular τ, by making a change of variables with a dilation. Also, the question is essentially the same for all n, because the n-dimensional integral reduces to the nth power of 1-dimensional integrals.

Because $H_\tau(1) \equiv 1$, $H_\tau(f)(x)$ is really an average of values of $f(y)$. This average is concentrated around x at the scale of $\sqrt{\tau}$ because of the decay of the Gaussian kernel of H_τ. It is not difficult to show, for instance, that

$$(3)\quad \sup_{x\in\mathbf{R}^n} |f(x) - H_\tau(f)(x)| = O(\tau^{\alpha/2})$$

when f is Lipschitz of order α.

The family of operators H_τ, $\tau > 0$, is an example of an *approximation to the identity*, as in [19], [20], [21]. In this case the kernels are especially nice, with a lot of symmetry. In particular, it turns out that the H_τ's form a semigroup, which is to say that $H_\sigma \circ H_\tau = H_{\sigma+\tau}$ for all $\sigma, \tau > 0$. In fact, $H_\tau(f)(x)$ satisfies the heat equation

$$\frac{\partial}{\partial\tau} H_\tau(f)(x) = \Delta H_\tau(f)(x),$$

where Δ denotes the usual Laplace operator on $\mathbf{R}^n$,

$$\Delta = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}.$$

Even if f is not smooth, $H_\tau(f)(x)$ is smooth in x and τ because of the smoothness of the kernel of H_τ.

Note that in the context of (1), the t parameter is incorporated into the metric space, while τ here is viewed as a kind of external parameter.

On a general space of homogeneous type, one can try to define approximations to the identity like this with as much symmetry as the space allows. It can be shown that there are always approximations to the identity with suitable localization and other nice properties in terms of basic analysis, along the lines of (3), for instance. Some basic aspects of this are reviewed in [18]. One may be interested in actual semigroups, perhaps with a discrete parameter running through the nonnegative integers rather than a continuous parameter τ as above, and this has been studied in a number of situations. A number of results along these lines are discussed in [11], and two recent papers related to this are [6], [7].

The Unit Ball in $\mathbf{C}^n$

Fix a positive integer n again, and let B_n denote the unit ball in $\mathbf{C}^n$, i.e., the set of z in $\mathbf{C}^n$ such that $|z| < 1$. Here $|z|$ denotes the usual Euclidean length of z, given by

$$|z|^2 = \sum_{j=1}^n |z_j|^2,$$

where z_j denotes the jth coordinate of z. Let us write Σ_n for the boundary of B_n, which is to say the set of z in $\mathbf{C}^n$ such that $|z| = 1$. If $f(z)$ is an integrable function on Σ_n, consider the function $F(w)$ on B_n defined by

$$(4)\quad F(w) = \int_{\Sigma_n} (1 - \langle w, z\rangle)^{-n} f(z)\, dA(z),$$

where

$$\langle w, z\rangle = \sum_{j=1}^n w_j\, \overline{z_j}$$

and $dA(z)$ denotes the element of surface integration on Σ_n, normalized so that the total area of Σ_n is equal to 1. Notice that $|\langle w, z\rangle| \le |w| < 1$ when $w \in B_n$ and $z \in \Sigma_n$, by the Cauchy–Schwarz inequality, so the integral in (4) makes sense. One calls $F(w)$ the *Cauchy–Szegő integral* of $f(z)$, and it is not hard to see that $F(w)$ is a holomorphic function of w, because the *Cauchy–Szegő kernel*

$$(5)\quad S_{B_n}(w, z) = (1 - \langle w, z\rangle)^{-n}$$

is holomorphic in w. It also turns out that if $F(w)$ is a holomorphic function on B_n with boundary values $f(z)$ in an appropriate sense, then $F(w)$ is reproduced from $f(z)$ by (4). When $n = 1$, this is a version of the Cauchy integral formula in one complex variable.

A basic property of the Cauchy–Szegő projection is that

$$(6)\quad \sup_{0<r<1} \int_{\Sigma_n} |F(rw)|^2\, dA(w) \le \int_{\Sigma_n} |f(z)|^2\, dA(z)$$

when f is in $L^2(\Sigma_n, dA)$ and F is as in (4). In other words, the Cauchy–Szegő integral defines an *orthogonal projection* of $L^2(\Sigma_n, dA)$ onto the subspace of functions which are boundary values of holomorphic functions on B_n. Indeed, the reproducing property mentioned in the previous paragraph shows that the mapping from f to the boundary values of F is a projection onto the subspace of functions which are boundary values of holomorphic functions on B_n. One can verify from the explicit expression that this projection is self-adjoint and hence is an orthogonal projection.

What about estimates in terms of L^p norms in place of L^2 norms? In other words, is there an inequality of the form

$$\sup_{0<r<1} \int_{\Sigma_n} |F(rw)|^p \, dA(w) \leq C(p) \int_{\Sigma_n} |f(z)|^p \, dA(z) \tag{7}$$

for all f in $L^p(\Sigma_n, dA)$, where $C(p)$ is a positive real number? Although one can take $C(2) = 1$, this does not work for other p. For $p = 1$, or for an analogous inequality involving essential suprema when $p = \infty$, such an estimate does not hold, but there are substitutes, and we shall say a bit about this in a moment. When $1 < p < \infty$, an estimate of this kind does hold. This goes back to famous work of Marcel Riesz when $n = 1$, and for $n > 1$ it was originally shown by Korányi and Vági. By now there are many results concerning questions like this, as explained in [20].

This kind of L^p estimate looks exactly like the type of issue which is addressed by singular integral theory, as in the real-variable methods of Calderón and Zygmund. Namely, there is already an L^2 estimate (6), and the Cauchy–Szegő kernel (5) is known and looks nice. The L^p estimates (7) do fall into the Calderón–Zygmund framework when $n = 1$, but not exactly when $n > 1$, because the kernel does not fit so well with Euclidean geometry. However, there is a different geometry on Σ_n with which it fits very well and which corresponds to a space of homogeneous type.

Actually, one can pretty much read the geometry off from the kernel in such a way that the two are then compatible. One can also describe the geometry in sub-Riemannian terms by saying that the distance between two points should be the infimum of the lengths of the paths in Σ_n which join the two points, subject to the constraint that the paths remain tangent to the complex subspaces of the tangent spaces to Σ_n. At any point p in Σ_n, the ordinary tangent space to Σ_n is a real plane with real dimension $2n - 1$, and it contains a complex plane of complex dimension $n - 1$, which is to say real dimension $2n - 2$. It turns out that the sub-Riemannian geometry just described is compatible with the usual topology on Σ_n but is quite different geometrically. Nonetheless, it defines a space of homogeneous type, and the Calderón–Zygmund methods apply to those in general, as in [3], [4]. It should be mentioned that the results of Korányi and Vági, as well as those of Frank Jones for the heat operator and singular integrals associated to it, were established before the development of the notion of spaces of homogeneous type and provide important examples of this notion.

When $n = 1$, there is a simple "Cayley transform" which permits one to move between the unit disk in $\mathbf{C}$ and the upper half-plane, which is the set of $z \in \mathbf{C}$ whose imaginary part is positive, through a holomorphic change of variables. There is an analogous transform when $n > 1$ for moving between the unit ball B_n and an "upper half-space", but now the latter is more complicated and has curved boundary. When $n = 1$, the boundary of the upper half-plane can be identified with the real line, and harmonic analysis on the real line is connected to various aspects of complex analysis, such as the corresponding Cauchy-Szegő projection, just as harmonic analysis on the unit circle is connected to complex analysis on the unit disk. For $n > 1$ it is natural to identify the boundary of the upper half-space mentioned above, not with an ordinary Euclidean space, but with the Heisenberg group of the appropriate dimension. Harmonic analysis on the Heisenberg group is then connected to complex analysis on the upper half-space, including the Cauchy-Szegő projection there. See [20] for more information.

In addition to L^p estimates for $1 < p < \infty$, there is a weak-type inequality for $p = 1$; Hardy space results for $p = 1$, or even $0 < p < 1$; and estimates in terms of bounded mean oscillation instead of L^∞ norms. Note that the definition of bounded mean oscillation here uses the special geometry on Σ_n rather than the ordinary Euclidean geometry, which would be appropriate for classical singular integral operators. There are results as well for $\mathrm{Lip}_\alpha(\Sigma_n)$, $0 < \alpha < 1$, which should also be interpreted using the sub-Riemannian geometry on Σ_n in place of the ordinary geometry, just as there are well-known results for classical singular integral operators and Lip_α spaces defined in terms of Euclidean geometry. In both the classical situation and this case, adjustments should be made for $\alpha = 1$, and one can deal with a variety of other function spaces as well. Concerning these various topics see [3], [4], [12], [19], [20].

Odd Kernels for Singular Integral Operators

The *Hilbert transform* is the linear operator acting on functions on the line defined by

$$H(f)(x) = \frac{1}{\pi} \, p.v. \int_{\mathbf{R}} \frac{1}{x - y} f(y) \, dy,$$

and the *Riesz transforms* are the linear operators acting on functions on $\mathbf{R}^n$ defined by

$$R_j(f)(x) = c_n\, p.v. \int_{\mathbf{R}^n} \frac{x_j - y_j}{|x - y|^{n+1}} f(y)\, dy,$$

$1 \le j \le n$, where c_n is the constant

$$\frac{\Gamma((n+1)/2)}{\pi^{(n+1)/2}}.$$

It is not hard to show that these principal value integrals exist when f is Lipschitz of some positive order and has compact support, for instance. These are the most fundamental examples of singular integral operators, and among their important properties is that they define bounded linear operators on L^p for $1 < p < \infty$. See [19], [21].

The Hilbert and Riesz transforms are singular integral operators of convolution type, which basically means that their kernels are of the form $k(x - y)$. On a general space of homogeneous type M, one can look at singular integral operators of the form

$$T(f)(x) = \int_M k(x, y) f(y)\, dy,$$

where the integral again involves some kind of principal values and where the kernel $k(x, y)$ satisfies size and smoothness conditions analogous to those of the Hilbert and Riesz transforms. Depending on the circumstances, the kernel might have additional symmetry or structure related to that of the underlying space M.

There are interesting ways in which the Hilbert and Riesz transforms take directions in the underlying Euclidean space into account, and a question that keeps bothering me is, what are reasonable versions of this in other situations? As far as I am concerned, the Cauchy–Szegő projections are also very interesting in this way, and in this regard one might like their counterparts on the corresponding upper half-spaces in $\mathbf{C}^n$, as on p. 536 of [20].

A basic feature of the kernels of the Hilbert and Riesz transforms is that they are odd, which is to say that they are of the form $k(x - y)$ where $k(-w) = -k(w)$. In general one can consider kernels $k(x, y)$ which are antisymmetric, so that

$$k(x, y) = -k(y, x).$$

Jean-Lin Journé once explained to me how this simple condition already has nice properties, although it is not as special as when $k(x, y)$ is something like a convolution kernel.

I think that it would be very interesting to have examples of singular integral operators in other contexts which are more like the Hilbert and Riesz transforms. This could involve some kind of reflections on the space about each point. In any case, there is a lot of room for interactions between kernels of singular integral operators and the structure of the underlying spaces.

References

[1] L. Ambrosio and F. Serra Cassano, eds., *Lecture Notes on Analysis in Metric Spaces*, Scuola Normale Superiore, Pisa, 2000.

[2] L. Ambrosio and P. Tilli, *Selected Topics on "Analysis in Metric Spaces"*, Scuola Normale Superiore, Pisa, 2000.

[3] R. Coifman and G. Weiss, *Analyse Harmonique Non-Commutative sur Certains Espaces Homogènes*, Lecture Notes in Math., vol. 242, Springer-Verlag, 1971.

[4] ____ , Extensions of Hardy spaces and their use in analysis, *Bull. Amer. Math. Soc.* **83** (1977), 569–645.

[5] M. Coornaert, Mesures de Patterson–Sullivan sur le bord d'un espace hyperbolique au sens de Gromov, *Pacific J. Math.* **159** (1993), 241–270.

[6] T. Coulhon, Off-diagonal heat kernel lower bounds without Poincaré, preprint, 2002.

[7] A. Grigoryan, J. Hu, and K. Lau, Heat kernels on metric-measure spaces and an application to semi-linear elliptic equations, *Trans. Amer. Math. Soc.*, to appear.

[8] M. Gromov, Hyperbolic groups, *Essays in Group Theory* (S. Gersten, ed.), Math. Sci. Res. Inst. Publ., vol. 8, Springer-Verlag, 1987, pp. 75–263.

[9] M. Gromov et al., *Metric Structures for Riemannian and Non-Riemannian Spaces*, Birkhäuser, 1999.

[10] J. Heinonen, *Lectures on Analysis on Metric Spaces*, Springer-Verlag, 2001.

[11] J. Kigami, *Analysis on Fractals*, Cambridge University Press, 2001.

[12] S. Krantz, *Geometric Analysis and Function Spaces*, CBMS Regional Conf. Ser. Math., vol. 81, Amer. Math. Soc., 1993.

[13] P. Pansu, Dimension conforme et sphère à l'infini des variétés à courbure négative, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **14** (1989), 177–212.

[14] M. Picardello and W. Woess, eds., *Random Walks and Discrete Potential Theory*, Cambridge University Press, 1999.

[15] N. Saldanha and C. Tomei, Spectra of regular polytopes, *Discrete Comput. Geom.* **7** (1992), 403–414.

[16] N. Saldanha, C. Tomei, M. Casarin, and D. Romualdo, Spaces of domino tilings, *Discrete Comput. Geom.* **14** (1995), 207–233.

[17] S. Semmes, Noyaux de la chaleur, marches aléatoires, analyses sur les variétés et les graphes, *Gaz. Math.* **95** (January, 2003).

[18] ____ , Happy fractals and some aspects of analysis on metric spaces, submitted to *Publ. Mat.*

[19] E. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton University Press, 1970.

[20] ____ , *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.

[21] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[22] R. Strichartz, Analysis on fractals, *Notices Amer. Math. Soc.* **46** (1999), 1199–1208.

[23] F. Thayer, Nonstandard analysis of graphs, *Houston J. Math.*, to appear.

[24] C. Tomei and T. Vieira, The kernel of the adjacency matrix of a rectangular mesh, *Discrete Comput. Geom.* **28** (2002), 411–425.

[25] N. Varopoulos, L. Saloff-Coste, and T. Coulhon, *Analysis and Geometry on Groups*, Cambridge University Press, 1992.