A Riemann surface of genus 1 is homeomorphic to the torus $T = S^1 \times S^1$. Therefore, a choice of a point to be the origin determines a group structure on the Riemann surface. An elliptic curve is a Riemann surface of genus 1 together with a choice of origin for the group structure. Although all elliptic curves are homeomorphic to the topological group $S^1 \times S^1$, they may have nonisomorphic complex structures. A natural question, called the problem of moduli, is to describe the space of all possible isomorphism classes of objects of a certain type.

In this article we discuss this question for elliptic curves. To an extent this is true: the isomorphism class of an elliptic curve is determined by a single complex number. A natural property and cannot be considered the true moduli space of elliptic curves.

A family of elliptic curves over a base space $B$ is a fibration $X \to B$ with a section $O : B \to X$ such that for every $b \in B$ the fiber $\pi^{-1}(b)$ is an elliptic curve with origin $O(b)$. Given a family of elliptic curves $X \to B$, define a classifying map $j_B : B \to \mathbb{C}$ by $b \mapsto j(\pi^{-1}(b))$. Because $\mathbb{C}$ is the coarse moduli space, $j_B(b) = j_B(b')$ if and only if the fibers $\pi^{-1}(b)$ and $\pi^{-1}(b')$ are isomorphic elliptic curves. Motivated by the concept of classifying space in topology, we require that a moduli space have a universal family. This means that if $\mathbb{C}$ were the moduli space of elliptic curves, there would exist a family of elliptic curves $E \to C$ such that every family would be obtained by pulling back the universal family via the map $j_B : B \to \mathbb{C}$. However, since every elliptic curve has an involution, there are nontrivial families of elliptic curves $X \to B$ such that $\pi^{-1}(b) \approx E_0$ for all $b \in B$, where $E_0$ is a fixed elliptic curve. (Such a family is called isotrivial.) The classifying map $j_B : B \to \mathbb{C}$ is the constant map $b \mapsto j(E_0)$. This contradicts the existence of a universal family, because the classifying map $B \to \mathbb{C}$ associated to the trivial family $E_0 \times B \to B$ is also constant.

To obtain the moduli space of elliptic curves, we must define a new concept, that of a stack. The stack of elliptic curves, $\mathcal{M}$, is a category. Its objects are families of elliptic curves, and a morphism $(X' \to B') \to (X \to B)$ is a pair of maps $X' \to X$, $B' \to B$ satisfying two conditions:

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X' \overset{\pi'}{\twoheadrightarrow} X
1. The diagram \( \pi' \downarrow \vdash X \)

\( B' \overset{f}{\rightarrow} B \)
2. \( X' \) is isomorphic to the pullback of \( X \) via the map \( B' \overset{\pi'}{\rightarrow} B \).

(Commutative diagrams satisfying condition (2) are called cartesian.) The subcategory of \( \mathcal{M} \) corresponding to families over a fixed base \( B \) is called the fiber over \( B \). Condition (2) says that the fibers of \( \mathcal{M} \) are groupoids, that is, categories where all morphisms are isomorphisms. More generally, \( \mathcal{M} \) is an example of an algebraic stack. An algebraic stack is a category fibered in groupoids which has a smooth covering by an affine variety, in a way which we explain below.

While this definition may look strange, we will see that there is a universal family of elliptic curves over the category \( \mathcal{M} \). For any variety \( B \) we can construct a similar category \( \mathcal{B} \): the objects are maps of varieties \( T \overset{f}{\rightarrow} B \), and a morphism \( (T' \overset{f'}{\rightarrow} B) \rightarrow (T \overset{f}{\rightarrow} B) \) is a map \( T' \overset{f}{\rightarrow} T \) such that \( f' \circ f \). It is relatively easy to show that the category \( \mathcal{B} \) determines \( B \), so we can identify \( B \) and \( \mathcal{B} \). To give a family of curves \( X \rightarrow B \) is equivalent to giving a functor (map of categories) \( \mathcal{B} \rightarrow \mathcal{M} \). Let \( C \) be the category whose objects are families of elliptic curves with a (nowhere zero) section and morphisms defined as for \( \mathcal{M} \). Forgetting the section defines a functor \( C \rightarrow \mathcal{M} \). For any family of elliptic curves \( X \rightarrow B \), the pullback of \( C \) via the corresponding map \( B \rightarrow \mathcal{M} \) is \( \mathcal{X} \). Thus, \( \mathcal{M} \) is the moduli space of elliptic curves and \( C \rightarrow \mathcal{M} \) is the universal family.

Finally, let us see what it means to say that \( \mathcal{M} \) has a smooth cover by an affine variety. Consider the Legendre family of elliptic curves \( y^2 = x(x-1)(x-\Lambda) \), where \( \Lambda \) varies in \( U = \mathbb{C} \setminus \{0,1\} \). The corresponding map \( U \rightarrow \mathcal{M} \) is a smooth cover in the following sense: given a map \( B \rightarrow \mathcal{M} \), the map \( U \rightarrow \mathcal{M} \) pulls back to a 12:1 branched covering \( B \rightarrow U \), which is a smooth covering by an affine variety.