Mathematics in a Small Place: Notes on the Mathematics of Romania and Bulgaria

Mark Saul

The Balkan states are proverbially unhappy. Buffeted by history, torn by ethnic and religious strife, mismanaged by unrepresentative or inefficient governments, they sit astride one of the fault lines of history, the division between Eastern and Western Christianity, between Asian and European empires, at the outer extremes of both continents. And yet the people of the Balkans flourish, making signal contributions to European and world culture. On two recent trips to the Balkans, I found much that we might learn from their view of mathematics as part of their cultural heritage.

It has been said that Greece is a poor country where there are many wealthy people, while Romania is a rich country where there are many poor people. For Romania does not prosper. Its economy proceeds on a microscopic scale. With 30,000 lei to the dollar, there is little a traveler would want to buy that would cost as much as $20, except a day's food and lodging.

I came in August 2001 to work with Balkan mathematicians on the interface between research mathematics and the K–12 curriculum, an area much better worked out in that part of the world than it is in the United States. The group met in Sinaia, a resort town stuffed into a mountain pass between Wallachia and Transylvania, two of the three medieval kingdoms that make up the modern state of Romania. (The third, Moldavia, we did not visit.)

Upon first arriving in Romania, I found myself sitting in a circle with a number of mathematicians. Titu Andreescu played the role of host. A gifted problem solver, he fled Romania during the 1989 revolution. In the United States he first taught high school and now serves as the coach of the U.S. International Mathematical Olympiad team.

Around the circle the talk was about old friends and new problems. It flowed back and forth from Romanian to English. Romanian, as the name suggests, is a Romance language, derived from Latin, and from time to time I could catch a few familiar words. And I could easily read the many problem books and elementary expositions that are a part of the tradition of Eastern European education. The circle included high school teachers, university professors, and researchers, and the talk flowed uninterrupted, without the issues of turf or rank that would inevitably come up in such a group in America.

Later, one of our sessions was devoted to quadratic equations, and I began discussing these with Ivan Tonov, a Bulgarian mathematician. Among the most tedious topics I have to teach in precalculus is the solution of simultaneous quadratic equations. The students need fluency in solving these so that they can find the intersection points of two conics in a calculus problem. Such questions are a staple of the Advanced Placement examination, which these students will take in about a year. The work is tedious, because there are few new concepts to be mastered, only accuracy in algebraic computation. Or so I thought until my conversation with Tonov. We looked at the problem and immediately saw that it could not be solved in general by elementary means. Eliminating one unknown, in the general case, leads to a quartic equation in the

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other unknown, whose solution is beyond the scope of what we teach in high school. So we must content ourselves with special cases. One special case that the texts unfailingly provide is when both equations lack linear terms. Then we can solve for x^2 and y^2 as variables themselves. So what are some other special cases? With this question the problem immediately became interesting, and I started looking forward to getting to the topic in the classroom.

Another special case occurs when one of the equations is homogeneous in the two variables. Then we can solve this equation for the ratio of the unknowns, express one as a multiple of the other, and substitute in the other equation to obtain a quadratic. We explored this a bit. A new question immediately arises. Suppose for such a system of equations that we eliminated one of the unknowns to get a quartic? What would be special about this quartic that would allow us to solve it? This general question launches us well into a discussion of Galois theory, which quickly goes beyond the usual high school curriculum. The interesting questions then become: What slices of Galois theory can we serve up to high school students? What, in the general situation, might be interesting to show high school teachers? How can we use all this to prepare some students, even a very few, for more advanced work? Or even lure them into more advanced work, using their natural curiosity? We didn't have time for all this, but now I look forward to getting to this topic in the classroom and also to discussing it with interested mathematicians.

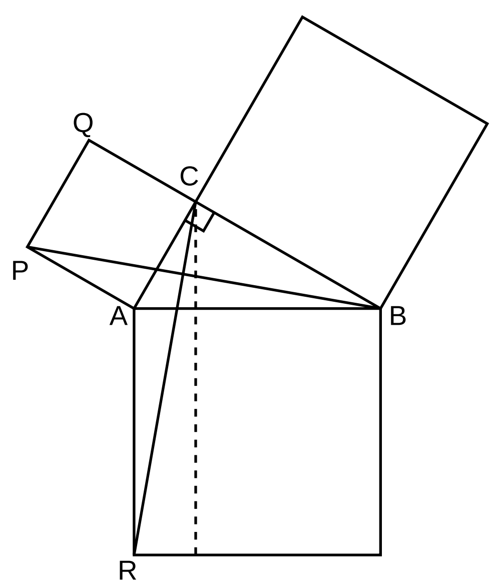
The point of our voyage to Romania, for me, was that in Eastern Europe I could find such interested mathematicians. In the United States such dialogue is very rare, and tends to be dominated by one side or the other. In a country like Tonov's Bulgaria, with everyone in the mathematical community crowded together, there is less room for specialization of interests, for carving out one's own territory, for building bridges between those working in different branches of the mathematical professions.

Szeged is the last big town in Hungary before you cross the Romanian border, for years one of the hottest in Europe. There is a sizeable Hungarian minority in Romania, and Transylvania, the largest of the three kingdoms that united to form present-day Romania, was Hungarian territory until 1918. So Szeged is almost in the Balkans, and Jozsef Kosztolanyi teaches there. He told me how he teaches high school, undergraduate courses, and graduate courses for teachers, and I immediately reacted with envy. Kosztolanyi has in one week the experiences of three American careers. In the United States we are too often limited by the institutions we work in.

"I got interested in mathematics very early. The Hungarian tradition in education differs from the Romanian in that we go deeper, not further. We don't place an emphasis on calculus in the high school years, nor on earlier algebra. Rather, we look for ways to get students to make better use of what they already know. We spend time developing their cognitive abilities."

Curious about this last point, I pressed him. Kosztolanyi replied with a reminiscence of his own education: "I had very good teachers in high school. It was a special school for mathematics. We were encouraged to reinvent for ourselves some of the classic results. For example, one teacher had us discover Euclid's proof of the Pythagorean Theorem."

Again I pressed. This beautiful proof is not at all intuitive. I had often taught it. I knew several ways to make the argument easy to follow. A close reading of Euclid himself tells us how to imbed the proof in a study of a more general method, that of the transformation of areas. But I had never succeeded in getting students to discover it for themselves. How is this done? "Well," explained Kosztolanyi, "we crept up on it. Aside from the various area formulas, which we knew well, the proof depends on the congruence of two particular triangles, which in turn can only be distinguished if we draw in certain lines." Kosztolanyi and I were enjoying a stroll through the park outside a palace of the former Romanian ruling family. But both of us had wrestled long enough with this proof to be able to speak without drawing a diagram. If we had drawn one, Kosztolanyi would be pointing to lines BP and CR in the figure. The congruent triangles in question are PAB and CAR.



"It's not easy to motivate the drawing of those lines. What my teacher did was to ask us to describe the diagram and to find equal elements. We could easily see that the sides of the various squares were equal, but the key insight—which some of us got—was to see that angles PAB and CAR are equal.

Once we saw that, we were able to distinguish the congruent triangles." And so I learned not just about Hungarian mathematics but about something I could use in my own teaching.

Florica Banu is a gifted teacher of junior high school students, so gifted that she has been working for the past year with the Romanian Ministry of Education to construct examinations in mathematics. I asked her how she first discovered mathematics. "I enjoyed mathematics very early," she replied. "But my mother discouraged me from studying it in college. 'Mathematics is not for silly girls,' she said. So I started studying physical education. Only I somehow always wanted to get back to mathematics. And after several years, I did. I enjoyed learning it, and now I enjoy teaching it."

No country has an exemplary record of involving girls in the study of mathematics. But the problem is coming to be recognized, and in Bulgaria I found some indication that the situation is improving. Jordan Tabov is a Bulgarian mathematician who has long been involved in various competitions, including the International Mathematical Olympiad. He told me, "We often have girls on our Olympiad team. Not only do the girls compete, but they often score very high. Once we asked a number of girls about the possibility of having a separate training camp for them, as we do for various sports. They did not want this, and some were even offended."

Petar Kenderov told me a bit more about this subject. Kenderov is the chair of the Department of Operations Research at the Bulgarian Mathematical Institute and has chaired numerous prestigious international committees. "Our special mathematical schools help a lot," Kenderov added. "They begin at age thirteen, a time when girls often show higher achievement than boys. Once they enter these schools, they are safe. Their talent is valued and nurtured."

Back in Romania, I missed one important encounter. It was related to me by Gail Richardson. Richardson works with Best Practices in Education, a not-for-profit group that explores practices abroad that we might use in the United States. The meeting I attended was one of a series arranged by this group, intended to bring teachers and mathematicians together, both American and Balkan. One of the Romanians at the meeting, a small man in a print shirt and suspenders, spoke no English. It was explained that he was the principal of the local Michail Cantacuzino High School. One day he invited the Americans to visit his school. Later, Richardson reported to me on the visit.

"Adrian spoke through a translator," she explained, "who was a former student. Adrian Ghioca has spent his whole working life in this school, first as a teacher, then as its principal. It was originally attached to a local factory, which closed down shortly after the Ceaușescu regime was toppled. At that point it was not clear what role the school ought to play. Adrian built it into a strong school, with an emphasis on mathematics and the sciences. Now the education it provides is highly coveted, and the school takes in boarding students from nearby regions.

"Adrian showed us the classrooms. Our lack of a common language did not prevent us from perceiving his passion for education and his pride in its achievements. Our translator, his former student, had tears in her eyes as she repeated his words. Pretty soon many of us did too. And this was even before he showed us the twenty-three books he had written on various areas of education."

We knew Adrian Ghioca only as a quiet chain-smoking man who always wore suspenders and who spoke no English. But now we were privileged to have seen the miracles this man had wrought in a tiny crack in the mountains of Transylvania.

One day Mircea Becheanu and I took a hike up one of the mountains girding the town of Sinaia. The exercise was not strenuous, and we talked as we climbed. The conversation ran to teaching. Becheanu is a professor of mathematics (algebra) at the University of Bucharest. He is deputy chair of the Romanian Mathematical Society and has done significant mathematical research. Like many of his colleagues, he is also active in education, editing the society's journal for teachers. Becheanu has taught all his life, including a stint building a mathematics department in Zaire. "Sometimes we lead with intuition," said Becheanu, "but sometimes a formal setting is clearer to the student. I studied tensor analysis for two years at the university, and it wasn't until the concept was made formal that I understood it." We had been talking not about tensor analysis, but about algebra and geometry in high school and the role of intuition in learning. Becheanu saw nothing remarkable in comparing high school algebra to tensor analysis: he had learned from teaching both.

Becheanu continued his remarks: "The formal tradition is very strong in Romania. It is the more or less conscious influence of Bourbaki. Sometimes our texts are too formal, but we also have had the Russian influence, after World War II, and the two schools of mathematics have blended here."

Romania has always felt an affinity for the French. Travelers to Bucharest remark on the resemblance of its architecture to that of Paris. Wedged in between the Turkish and the Russian empires, with their backs to the Germans and Austrians, the Romanians are used to the grindstones of history. France was the nearest European power that seemed not to threaten the country's sovereignty. And so Romanian culture has often followed the French. The Bourbaki school made a lasting impression here, both on the research community and on the schools. "After the [Second World] War, we had the Russian model," explained Becheanu. "The country was turned eastward. And so the formalism of Bourbaki seemed at that time to be something patriotic, an act of resistance to the satellite status of the country. Now that the political climate is looser, we are more free to choose the influence we think best."

Geography is destiny, some say, and both Romania and Bulgaria have had similar destinies—and similar geographies. But the countries are really very different. Romania is more than twice as large as Bulgaria, both in area and in population, and has always been wealthier. The Bulgarians, however, have had a stronger sense of nationality. They have had this since the Middle Ages, when the first Bulgarian kingdom won its independence from Byzantium. Bulgaria was also the strongest national state in the Balkans during the period just after the Turkish occupation. Romanian national feeling is much newer: there was no strong medieval kingdom, and the country coalesced out of three provinces as the Turkish Empire crumbled.

Bulgarian history came alive on a visit one spring. Jenny Sendova, a researcher at the Bulgarian Academy of Sciences, had invited me to help construct a summer program for high school students. It was to be modeled on the Research Science Institute, a summer program run by the Center for Excellence in Education and the Massachusetts Institute of Technology. Students are provided internships with working scientists and mathematicians. Bulgarian mathematicians and scientists are used to working with young people, so an internship program is a natural development for the country. I was there to help develop internship sites, and I had the delightful job of listening to various researchers describe their work.

"We have a great national treasure," Milena Dobreva told me. "Our libraries and monasteries are filled with medieval manuscripts. I am creating a searchable database so that scholars can work with them more easily. This will be more than just a list of the manuscripts. The database will contain searchable, digital copies."

Being closest to Constantinople, the Bulgarians were among the first Slavs to be Christianized and the first to have a written language. Bulgarians are often taken aback when the Cyrillic alphabet is referred to as the "Russian" alphabet, for it was invented by Bulgarian monks. For centuries, Old Church Slavonic, an archaic variant of Bulgarian, was the literary language of the Slavs. And it was in this language that the earliest Slavic traditions of Christianity developed. One of the difficulties Dobreva faces is that the script is not linear: letters occur over, under, or inside other letters, and the letters themselves assume different forms. All this information will be important to researchers and will have to be preserved in digital form.

"I am looking at tools to provide for researchers who use these manuscripts," explained Dobreva. "They often have to trace the source of the works. The scribes would make errors or create variants as they transcribed the texts. Sometimes they would insert a letter or a word, or even a whole passage. Sometimes they would delete a word or passage. And sometimes they would replace one whole section with another. In tracing the origin of a manuscript, we have to compare the sequence in one text with the sequence in another."

"So," I commented, "it sounds like what the molecular biologists are doing lately with sequences of nucleotides."

"Exactly," said Dobreva. "The algorithms used in nucleotide sequencing could be applied to the study of some aspects of medieval manuscripts." And so the generality of mathematical methods asserts itself, over thousands of years and thousands of miles.

"'Bulgaria? I think I know where that is. The capital is Bucharest, right?' That's what people sometimes ask me." Jenny Sendova and I were walking around the large church in the downtown section of the Bulgarian capital, which is, of course, Sofia, not Bucharest. The church memorializes the Russian army, which liberated the country from the Turks. Throughout my visit I found this fear of oblivion. Living in a small country, tucked away in a side pocket of Europe, it is not so much the fear of being stigmatized or stereotyped that one feels, as the fear of being forgotten, of being treated as inconsequential. And so every Bulgarian connection seems important: the heritage of Christianity and literacy, the wonderfully inventive folk music and folk dance, the individual mathematicians or artists who have from time to time contributed to Western culture.

I often think that Americans, for all their prosperity, and their robust culture, now emulated by the rest of the world, have oddly parallel fears. Ours is a new country. We don't have our own language or a recognizable national costume. Our country's name might refer to a dozen other entities if it weren't familiar as our own. We are stereotyped, not as quaint, but as callow. And we are too prominent, too "vanilla", on the world stage. We fear being young and inexperienced.

Petar Kenderov told me a bit about the traditions of mathematics in Bulgaria. "Kyril Popov (the name has various transliterations) was an early Bulgarian mathematician. A student of Poincaré, he worked in analysis. He was active and highly respected well into the 1950s. But more important than the French influence in Bulgaria was the Russian. After the 1917 revolution, many Russian intellectuals settled in Bulgaria, including several mathematicians. In a small country like ours, even two or three hundred remarkable people can influence the intellectual life. But young people, at that time, had to find their own way to mathematics.

"More recently, Ljubomir Iliev encouraged young people more directly to seek careers in mathematics. He started the Balkan Mathematical Society. While his own research was in complex analysis, Iliev worked to enhance development of all branches of the field, including computer science as a branch of mathematics. He encouraged and recruited younger people to work on these branches throughout the 1970s and 1980s. At this time a critical mass was formed of people interested in mathematics in this country. The International Mathematical Olympiad played a large role in this development.

"It was Iliev, on a visit to Russia, who first got a vision of how computers could play a role in the development of Bulgaria's economy. Ours was the first country in the Eastern Bloc to produce CDs and personal computers. It was no coincidence that the first olympiad in computer science was held in Bulgaria. Iliev even obtained exemptions from service in the army for young people, so that they could work with him. The selection process was quite rigorous."

Kenderov adds, a bit too modestly, "I was one of those chosen. At that time, the minister of education, Gancho Ganev, had a background in mathematics. He sent many students to Moscow. Bulgarian students formed a large contingent at Moscow State University. And almost no one in mathematics ever defected to the West. Ironically, it is now that mathematicians are leaving, especially in computer science.

"In Bulgaria, we have always recognized the role of teachers in identifying talent. Their job is not just to teach mathematics. The identification and support of talented students is their more difficult task. And this is the most vulnerable place in our system just now. The Cyril and Methodius Foundation [a charitable international organization named after the two brothers who brought literacy to Bulgaria, of which Kenderov is president] gives prizes to teachers with success in identifying and developing talented students."

While the cultures of the Balkans are old, the nations themselves are young, and political traditions are still evolving. I met one day with Blagovest Sendov at his office in the Central Laboratory for Parallel Computing. He comes here every morning before going off to assume his duties in the Bulgarian parliament. Sendov is an imposing figure, both physically and intellectually. Having enjoyed a robust career as a mathematician, he took up politics after the fall of Communism. He was elected to the parliament, then to its leadership as chairman. He is as deeply concerned about the fate of his country as he is about that of the mathematics it has produced. Concern marks his wide brow as he speaks. "We have trouble here. Our government is chaotic. Money is flowing out of the country at an alarming rate, and we have lost a large part of our irrigated cropland. I have tried to speak out about this situation, but some mistake my words for a call to return to the old system."

Sendov speaks with the intensity of a man used to solving hard problems, problems that one must wrestle with for long periods of time before subduing them. But even he seems weighed down by the problems of his country. It may take more than one lifetime to solve them. Sendov might have rested content with his achievements in mathematics. Or he could have emigrated to a Western university. But he stayed and is trying to use his problem-solving abilities to help his country. In fact, it is not that uncommon in small countries for people prominent in the sciences to engage also in political activity. With fewer human resources, sometimes active minds must do double duty.

I have written elsewhere[1] about Russian mathematics and how it played the role of a refuge from the totalitarian state for intellectually active people. The same situation held all over Eastern Europe. Perhaps now that these societies have burst open, mathematical minds are free to range over a larger domain of activity. On the other hand, it may be useful to think of the field of education as less differentiated than that of research. We need people who are skilled in introspection in a close examination of mathematical worlds. Sometimes these same people are also gifted in stimulating the thoughts of others—the quintessential act of teaching. And even if these talents reside in different people, it is important that these different people achieve a synergy of effort. I observed such a synergy in the Balkans.

That is, I found the Balkan mathematics community less balkanized than the American.

Acknowledgment

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[1] *M. Saul, Love Among the Ruins: The Education of High Ability Mathematics Students in the USSR,* Focus ***12** (February 1992).*