# An Introduction to Heisenberg Groups in Analysis and Geometry

*Stephen Semmes*

Heisenberg groups, in discrete and continuous versions, appear in many parts of mathematics, including Fourier analysis, several complex variables, geometry, and topology. In the present survey we shall not focus too much on any particular aspect, but try to give a kind of sampler. We begin with something which is, in effect, basic calculus.

**Commutators of Multiplication and Differentiation Operators**

Fix a positive integer $n$, and let $v \cdot w$ denote the usual dot product on $\mathbf{R}^n$, so that $v \cdot w = \sum_{j=1}^n v_j\, w_j$, where $v_j$ and $w_j$ denote the $j$th components of $v$ and $w$, respectively. If $v$ and $w$ are elements of $\mathbf{R}^n$ and $f$ is a function on $\mathbf{R}^n$, let us write $M_w(f)$ for the function on $\mathbf{R}^n$ defined by $M_w(f)(x) = (w \cdot x)\, f(x)$, and let us write $D_v(f)$ for the function on $\mathbf{R}^n$ which is the directional derivative of $f$ associated to $v$, i.e., $D_v(f)(x) = v \cdot \nabla f(x)$. We shall not dwell on differentiability issues here, and so the reader is invited to assume that the functions are sufficiently smooth as might be convenient. The usual Leibniz rule for differentiating products implies that

$$D_v(M_w(f)) - M_w(D_v(f)) = (v \cdot w) f. \tag{1}$$

*Stephen Semmes is professor of mathematics at Rice University. His email address is* `semmes@math.rice.edu`.

*The author is grateful to an unnamed reader for very helpful comments and suggestions.*

*Dedicated to Eli Stein and Guido Weiss.*

In other words, the commutator of the linear operators $M_w$ and $D_v$ is equal to a constant multiple of the identity operator, and that constant can be nonzero in general.

Note that the commutator of two real or complex matrices of finite rank can never be equal to a nonzero constant multiple of the identity matrix, because the trace of the commutator is automatically zero. This observation uses the standard identity that the trace of a product $AB$ is the same as the trace of the product $BA$. For linear operators on infinite-dimensional spaces, there are still results to the effect that the commutator of two *bounded* linear operators cannot be a nonzero constant multiple of the identity, as on pp. 350–1 of [18]. See also [5].

As is well known, the fact that the $M_w$'s and the $D_v$'s do not commute with each other in general is connected to the Heisenberg uncertainty principle in quantum mechanics.

**Some Groups of Linear Operators**

Let $n$ still be a fixed positive integer, and let us continue to consider linear operators acting on functions on $\mathbf{R}^n$. For each $v$ in $\mathbf{R}^n$, define the operator $T_v$ to be the operator of translation by $v$, so that

$$T_v(f)(x) = f(x - v).$$

Thus $T_v$ is the identity operator exactly when $v = 0$, and

$$T_v \circ T_{v'} = T_{v+v'}$$

for all $v, v' \in \mathbf{R}^n$. In other words, the family of operators of the form $T_v$, $v \in \mathbf{R}^n$, forms an abelian group which is isomorphic to $\mathbf{R}^n$ as a group under addition.

Fix a positive real number $\lambda$. For each $w$ in $\mathbf{R}^n$, define $U_w$ to be the operator of multiplication by $\exp(2\pi i\lambda\, w \cdot x)$, so that

$$U_w(f)(x) = \exp(2\pi i\,\lambda\, w \cdot x)\, f(x).$$

We should work with complex-valued functions now to accommodate this complex exponential. As before, $U_w$ is equal to the identity operator if and only if $w = 0$, and

$$U_w \circ U_{w'} = U_{w+w'}$$

for all $w, w' \in \mathbf{R}^n$. Thus the family of operators $U_w$, $w \in \mathbf{R}^n$, forms an abelian group which is also isomorphic to $\mathbf{R}^n$ under addition.

How do these two families of operators interact? What kind of group do they generate? To see this, we can compute as follows:

$$\begin{aligned} &T_v(U_w(f))(x) \\ &\quad = U_w(f)(x - v) \\ &\quad = \exp(2\pi i\,\lambda\, w \cdot (x - v))\, f(x - v) \\ &\quad = \exp(-2\pi i\,\lambda\, w \cdot v)\, \exp(2\pi i\,\lambda\, w \cdot x)\, T_v(f)(x) \\ &\quad = \exp(-2\pi i\,\lambda\, w \cdot v)\, U_w(T_v(f))(x). \end{aligned}$$

In other words, $T_v \circ U_w$ is equal to a constant multiple of $U_w \circ T_v$, where that constant is equal to $\exp(-2\pi i\,\lambda\, w \cdot v)$. This is equivalent to saying that the commutator $T_v \circ U_w \circ (T_v)^{-1} \circ (U_w)^{-1}$ is equal to a constant multiple of the identity, where that constant is equal to $\exp(-2\pi i\,\lambda\, w \cdot v)$. This is a kind of multiplicative version of the commutator equation (1).

Using this identity, one can check that the family of operators of the form $\alpha\, T_v \circ U_w$, with $\alpha \in \mathbf{C}$, $|\alpha| = 1$, and $v, w \in \mathbf{R}^n$, forms a group under composition. Note that the *Fourier transform* interchanges the roles of first-order directional derivatives and multiplication by linear functions, and also the roles of translation operators and multiplication by exponentials as above.

**Definition of the $n$th Heisenberg Group**

Let us define $H_n(\mathbf{R})$ as follows. First, as a set, $H_n(\mathbf{R})$ is equal to $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$, i.e., the set of ordered triples $(v, w, t)$, where $v$ and $w$ lie in $\mathbf{R}^n$ and $t$ lies in $\mathbf{R}$. Next, we define a binary operation on $H_n(\mathbf{R})$ by

$$\begin{aligned} (2)\quad (v, w, t) \bullet (v', w', t') \\ = (v + v', w + w', t + t' + v' \cdot w). \end{aligned}$$

One can verify that with respect to this operation, $H_n(\mathbf{R})$ becomes a group, with $(0, 0, 0)$ as the identity element, and $(-v, -w, v \cdot w - t)$ the inverse of $(v, w, t)$.

When $n = 1$, one can associate a triple $(v, w, t)$ to a $3 \times 3$ real matrix

$$(3)\qquad \begin{pmatrix} 1 & w & t \\ 0 & 1 & v \\ 0 & 0 & 1 \end{pmatrix},$$

and then the group operation (2) corresponds exactly to matrix multiplication. In general, $H_n(\mathbf{R})$ can be identified with a subgroup of the group of $(n + 2) \times (n + 2)$ real matrices with 1's along the diagonal and 0's below the diagonal.

Let $\lambda$ be a fixed positive real number, and let $T_v$ and $U_w$ be the linear operators on functions on $\mathbf{R}^n$ as before. Consider the mapping from $H_n(\mathbf{R})$ to linear operators defined by

$$(v, w, t) \mapsto \exp(2\pi i\,\lambda\, t)\, T_v \circ U_w.$$

One can verify that this mapping is a group homomorphism, which is to say that the group operation defined above for $H_n(\mathbf{R})$ corresponds to composition of linear operators. An element $(v, w, t)$ of $H_n(\mathbf{R})$ is mapped to the identity operator exactly when $v = w = 0$ and $\lambda\, t$ is an integer. Thus this homomorphism incorporates a periodicity in the $t$ variable but is otherwise injective.

Let us be more precise and think of these linear operators as acting on the Hilbert space $L^2(\mathbf{R}^n)$ of square-integrable functions on $\mathbf{R}^n$. These operators are all *unitary*, which is to say that they map $L^2(\mathbf{R}^n)$ onto itself and preserve the usual integral Hermitian scalar product there. Thus for each $\lambda > 0$ one gets a "unitary representation" of $H_n(\mathbf{R})$ on $L^2(\mathbf{R}^n)$.

In addition to the group structure on $H_n(\mathbf{R})$, there is a natural family of *dilations*. Namely, for each positive real number $r$, define a mapping $\delta_r$ from $H_n(\mathbf{R})$ to itself by

$$\delta_r(v, w, t) = (r\, v, r\, w, r^2\, t).$$

This is a one-to-one mapping of $H_n(\mathbf{R})$ onto itself which preserves the group structure on $H_n(\mathbf{R})$. Also,

$$\delta_r \circ \delta_{r'} = \delta_{rr'}$$

for all $r, r' > 0$, and $\delta_r$ is the identity transformation exactly when $r = 1$.

If $a = (v_0, w_0, t_0)$ is an element of $H_n(\mathbf{R})$, consider the mapping $L_a$ from $H_n(\mathbf{R})$ to itself defined by left-multiplication by $a$, so that

$$L_a(v, w, t) = (v_0, w_0, t_0) \bullet (v, w, t).$$

It is not difficult to verify that ordinary Euclidean volumes are preserved by $L_a$, which is to say that the volume of a measurable set $E$ is the same as the volume of the image $L_a(E)$ of $E$ under $L_a$. As in ordinary vector calculus, this statement is equivalent to saying that the Jacobian of the mapping $L_a$, which is the determinant of the differential of $L_a$, is equal to 1 everywhere on $H_n(\mathbf{R})$. Indeed, it is easy to see that the differential of $L_a$ can be written in terms of an upper-triangular matrix at each point, where the diagonal part of the matrix is the same as the identity matrix. For an upper-triangular matrix, the determinant is simply the product of the diagonal entries and hence is equal to 1 everywhere in this case.

In short, ordinary volume measure is invariant under left translations in the Heisenberg group. Similarly, it is invariant under right translations. It also behaves nicely under dilations; namely, the volume of $\delta_r(E)$ is equal to $r^{2n+2}$ times the volume of $E$ for all $r > 0$ and all measurable sets $E$ in $H_n(\mathbf{R})$. The number $2n+2$ is sometimes called the *homogeneous dimension* of $H_n(\mathbf{R})$ for this reason. By contrast, if $A$ is a measurable subset of $\mathbf{R}^k$ and one applies a standard dilation by a factor of $t > 0$ to $A$, then the volume of the result is $t^k$ times the volume of $A$ itself. Thus for ordinary Euclidean spaces the "homogeneous dimension" is the same as the vector-space dimension, which is the same as the topological dimension, while for $H_n(\mathbf{R})$ the homogeneous dimension is equal to the topological dimension plus 1.

### Discrete Versions of the Heisenberg Groups

Define $H_n(\mathbf{Z})$ to be the set of triples $(v, w, t)$, where $v$ and $w$ lie in $\mathbf{Z}^n$ and $t$ lies in $\mathbf{Z}$. It is easy to see that this is a subgroup of $H_n(\mathbf{R})$. It is a discrete version of the Heisenberg group.

In addition to being a group, $H_n(\mathbf{R})$ is a smooth manifold of dimension $2n+1$, and the group operation is smooth. The quotient $H_n(\mathbf{R})/H_n(\mathbf{Z})$ makes sense not only as a set of cosets, but also as a compact smooth manifold of dimension $2n+1$ without boundary. When $n = 1$, this manifold has dimension 3, and it is one of the basic building blocks for 3-manifolds discussed in [23].

Define $a_j$ in $H_n(\mathbf{Z})$ for $j = 1, \dots, n$ to be the triple $(v, w, t)$ such that all components of $v$ are equal to 0 except for the $j$th component, which is equal to 1, and such that $w = 0$ and $t = 0$. Similarly, define $b_j$ in $H_n(\mathbf{Z})$ for $j = 1, \dots, n$ to be the triple $(v, w, t)$ such that $v = 0$; all components of $w$ are equal to 0 except for the $j$th component, which is equal to 1; and $t = 0$. Define $c$ in $H_n(\mathbf{Z})$ to be the triple $(0, 0, 1)$. It is easy to see that the elements

$$a_1, \dots, a_n, b_1, \dots, b_n, c$$

generate $H_n(\mathbf{Z})$. Indeed, if $k$ and $l$ lie in $\mathbf{Z}^n$ and $m$ lies in $\mathbf{Z}$, then $(k, l, m)$ is the same as the product

$$a_1^{k_1} \cdots a_n^{k_n} b_1^{l_1} \cdots b_n^{l_n} c^m \tag{4}$$

in the group, where of course this expression is interpreted using the group operation.

For $i, j = 1, \dots, n$ we have that

$$a_i\, a_j = a_j\, a_i, \quad b_i\, b_j = b_j\, b_i$$

and

$$a_i\, c = c\, a_i, \quad b_j\, c = c\, b_j.$$

If $1 \le i, j \le n$ and $i \ne j$, then

$$a_i\, b_j = b_j\, a_i.$$

When $i = j$ we have in place of this

$$b_i\, a_i = a_i\, b_i\, c$$

for $i = 1, \dots, n$. These relations describe the group $H_n(\mathbf{Z})$ completely; every element of $H_n(\mathbf{Z})$ can be represented in a unique way in the form (4), and these relations are adequate to define the group operation in terms of this representation.

### Connections with Several Complex Variables

It will be convenient now to use a slightly different formulation of the $n$th Heisenberg group. Define $\widetilde{H}_n$ to be $\mathbf{C}^n \times \mathbf{R}$ equipped with the binary operation

$$(z, t) \diamond (z', t') = \left(z + z', t + t' + 2\,\mathrm{Im} \sum_{j=1}^{n} z_j\, \overline{z'_j}\right),$$

where $z_j$ and $z'_j$ denote the $j$th components of $z$ and $z'$, and $\overline{a}$ denotes the complex conjugate of a complex number $a$. One can check that this defines a group operation on $\widetilde{H}_n$, where $(0, 0)$ is the identity element and $(-z, -t)$ is the inverse of $(z, t)$. In fact $\widetilde{H}_n$ is isomorphic to $H_n(\mathbf{R})$, as one can see by associating to $(x, y, t)$ in $H_n(\mathbf{R})$ the element

$$(x, 0) \diamond (iy, 0) \diamond (0, t) \tag{5}$$

of $\widetilde{H}_n$. The correspondence $(x, y, t) \mapsto (x + i\,y, t)$ does not quite work, and (5) adds in a suitable correction.

Put

$$\mathcal{U}_{n+1} = \{(w, \sigma) \in \mathbf{C}^n \times \mathbf{C} : \mathrm{Im}\,\sigma > |w|^2\}, \tag{6}$$

where $|w|^2 = \sum_{j=1}^{n} |w_j|^2$, as usual. Thus

$$\partial\mathcal{U}_{n+1} = \{(w, \sigma) \in \mathbf{C}^n \times \mathbf{C} : \mathrm{Im}\,\sigma = |w|^2\}.$$

For each $(z, t)$ in $\widetilde{H}_n$, define the mapping $A_{(z,t)}$ on $\mathbf{C}^n \times \mathbf{C}$ by

$$A_{(z,t)}(w, \sigma) = \left(w + z, \sigma + t + i|z|^2 + 2i \sum_{j=1}^{n} w_j\, \overline{z_j}\right). \tag{7}$$

Because

$$|w + z|^2 = |w|^2 + 2\,\mathrm{Re} \sum_{j=1}^{n} w_j\, \overline{z_j} + |z|^2,$$

one can check that $A_{(z,t)}(w, \sigma) \in \mathcal{U}_{n+1}$ when $(w, \sigma) \in \mathcal{U}_{n+1}$ and $A_{(z,t)}(w, \sigma) \in \partial\mathcal{U}_{n+1}$ when $(w, \sigma) \in \partial\mathcal{U}_{n+1}$. Also,

$$
\begin{aligned}
&A_{(z,t)}(A_{(z',t')}(w,\sigma))\\
&= \Big(w + z' + z, \sigma + t' + t + i|z'|^2 + i|z|^2\\
&\qquad + 2i\sum_{j=1}^n w_j\,\overline{z_j'} + 2i\sum_{j=1}^n (w_j + z_j')\,\overline{z_j}\Big)\\
&= \Big(w + z' + z, \sigma + t' + t + i|z'|^2 + i|z|^2\\
&\qquad + 2i\sum_{j=1}^n w_j\,\overline{(z_j' + z_j)} + 2i\sum_{j=1}^n z_j'\,\overline{z_j}\Big)\\
&= \Big(w + z' + z, \sigma + t' + t + i|z' + z|^2\\
&\qquad + 2i\sum_{j=1}^n w_j\,\overline{(z_j' + z_j)} - 2\,\mathrm{Im}\sum_{j=1}^n z_j'\,\overline{z_j}\Big)\\
&= \Big(w + z + z', \sigma + t + t' + i|z + z'|^2\\
&\qquad + 2i\sum_{j=1}^n w_j\,\overline{(z_j + z_j')} + 2\,\mathrm{Im}\sum_{j=1}^n z_j\,\overline{z_j'}\Big)\\
&= A_{(z,t)\diamond(z',t')}(w,\sigma).
\end{aligned}
$$

In particular, $A_{(-z,-t)}$ is the inverse of $A_{(z,t)}$. It follows that $A_{(z,t)}$ defines a one-to-one mapping of $\mathcal{U}_{n+1}$ onto itself.

From the definition of $A_{(z,t)}$ it is clear that $A_{(z,t)}$ is a *holomorphic* mapping, and in fact a complex-affine mapping, since there are only $w$'s and $\sigma$'s in $A_{(z,t)}(w,\sigma)$ and none of their complex conjugates. Thus $A_{(z,t)}$ is in fact a *biholomorphic* mapping of $\mathcal{U}_{n+1}$ onto itself.

Define $\widetilde{\delta}_r : \widetilde{H}_n \to \widetilde{H}_n$ for $r > 0$ by

$$\widetilde{\delta}_r(z,t) = (r\,z, r^2\,t).$$

As before, $\widetilde{\delta}_r$ is a one-to-one mapping of $\widetilde{H}_n$ onto itself which preserves the group structure on $\widetilde{H}_n$, $\widetilde{\delta}_r \circ \widetilde{\delta}_{r'} = \widetilde{\delta}_{rr'}$ for all $r, r' > 0$, and $\widetilde{\delta}_r$ is the identity mapping on $\widetilde{H}_n$ exactly when $r = 1$. These dilations also correspond to mappings on $\overline{\mathcal{U}_{n+1}}$ in a natural way. Namely, define $\Delta_r : \overline{\mathcal{U}_{n+1}} \to \overline{\mathcal{U}_{n+1}}$ for $r > 0$ by

$$\Delta_r(w,\sigma) = (r\,w, r^2\,\sigma). \tag{8}$$

It is easy to see that $\Delta_r$ is a one-to-one mapping of $\overline{\mathcal{U}_{n+1}}$ onto itself which takes $\mathcal{U}_{n+1}$ to $\mathcal{U}_{n+1}$ and that $\Delta_r \circ \Delta_s = \Delta_{rs}$ for all $r, s > 0$ and $\Delta_1$ is the identity mapping. Furthermore, $\Delta_r$ is a biholomorphic mapping of $\mathcal{U}_{n+1}$ onto itself, and

$$\Delta_r(A_{(z,t)}(w,\sigma)) = A_{\widetilde{\delta}_r(z,t)}(\Delta_r(w,\sigma)).$$

A famous fact about $\mathcal{U}_{n+1}$ is that it is biholomorphically equivalent to the unit ball in $\mathbf{C}^{n+1}$. Thus the $A_{(z,t)}$'s and $\Delta_r$'s correspond to biholomorphic transformations on the unit ball in $\mathbf{C}^{n+1}$. They do not account for all of the biholomorphic transformations, though; basically what is missing are the complex-linear transformations on $\mathbf{C}^{n+1}$ which map the unit ball onto itself, which is the same as saying that they preserve the standard Hermitian inner product on $\mathbf{C}^{n+1}$.

### Tube Domains and Spaces of Holomorphic Functions on Them

Let $m$ be a positive integer, and let $B$ be a nonempty open subset of $\mathbf{R}^m$. The corresponding *tube domain* $T_B$ in $\mathbf{C}^m$ is defined by

$$T_B = \{z \in \mathbf{C}^m : z = x + i\,y,\ x \in \mathbf{R}^m, y \in B\}.$$

We shall be interested in complex-valued functions $F(z)$ on $T_B$ which are holomorphic, which is to say that $F(z)$ can be represented in a neighborhood of each point $w \in T_B$ by a power series that uses only factors of the form $(z_j - w_j)$ and not their complex conjugates. If $F(z)$ is a function on $T_B$ which admits a power series representation around each point $w \in T_B$ that uses both the factors $(z_j - w_j)$ and their complex conjugates, then $F(z)$ is said to be real-analytic. Of course real-analyticity can be defined directly on real Euclidean spaces, using standard power series expansions, and that definition is equivalent to this one for functions on open subsets of $\mathbf{C}^m$.

A real-analytic function vanishes on a neighborhood of some point if and only if the function and all of its derivatives are equal to 0 at that point. Using this, one can check that if a real-analytic function on a connected domain vanishes on a nonempty open subset, then the function vanishes everywhere on the domain. We shall make the standing assumption that the bases $B$ of our tube domains are connected, so that the corresponding tube domain is also connected.

Fix a point $y$ in $B$, and consider the subset $\mathbf{R}^m + i\,y$ of $T_B$ consisting of points of the form $x + i\,y$, $x \in \mathbf{R}^m$. If $F(z)$ is a holomorphic function on $T_B$, then $F(z)$ is uniquely determined by its restriction to $\mathbf{R}^m + i\,y$. This is equivalent to saying that if $F(z)$ is holomorphic on $T_B$ and vanishes on $\mathbf{R}^m + i\,y$, then $F(z)$ vanishes on all of $T_B$. Indeed, if $F(z)$ vanishes on $\mathbf{R}^m + i\,y$, then the derivatives of $F(z)$ in the $\mathbf{R}^m$ directions on $\mathbf{R}^m + i\,y$ vanish, and for a holomorphic function this implies that the derivatives in all directions vanish on $\mathbf{R}^m + i\,y$. In other words, the power series coefficients of $F(z)$ at points in $\mathbf{R}^m + i\,y$ all vanish, so that $F(z)$ vanishes on a neighborhood of $\mathbf{R}^m + i\,y$. In particular, $F(z)$ vanishes on a nonempty open subset of $T_B$ and hence on all of $T_B$, as in the previous paragraph.

Thus one can identify holomorphic functions on $T_B$ with a special class of functions on $\mathbf{R}^m + i\,y$. Just as the $m$th Heisenberg group acts by linear operators on functions on $\mathbf{R}^m$, as indicated before, it also acts on holomorphic functions on $T_B$. For each fixed $\xi$ in $\mathbf{R}^m$, the mapping $z \mapsto z - \xi$ takes $T_B$ onto itself, and this leads to the linear operator

$F(z) \mapsto F(z-\xi)$ acting on functions on $T_B$, which takes holomorphic functions to holomorphic functions. If we fix a positive real number $\lambda$, then for each $\eta$ in $\mathbf{R}^m$ we get a complex exponential

$$\exp\Big(2\pi i\lambda \sum_{j=1}^{m} z_j\,\eta_j\Big) \tag{9}$$

which defines a holomorphic function on all of $\mathbf{C}^m$. Multiplication by this exponential function defines a linear operator on functions on $T_B$, and it takes holomorphic functions to holomorphic functions, because the exponential function is holomorphic. Just as before, the translation operators and the operators of multiplication by these complex exponentials do not quite commute, the commutators between them being scalar multiples of the identity, and again we get homomorphic images of the $m$th Heisenberg group.

As in Chapter III of [21], one defines the Hardy space $\mathcal{H}^2(T_B)$ to be the vector space of holomorphic functions $F(z)$ on $T_B$ such that

$$\int_{\mathbf{R}^m} |F(x+i\,y)|^2\,dx \tag{10}$$

is finite for all $y \in B$ and in fact uniformly bounded. The norm $\|F\|_{\mathcal{H}^2(T_B)}$ is defined by

$$\|F\|^2_{\mathcal{H}^2(T_B)} = \sup_{y\in B}\int_{\mathbf{R}^m} |F(x+i\,y)|^2\,dx.$$

This space and its norm are preserved by the translation operators $F(z)\mapsto F(z-\xi)$, $\xi \in \mathbf{R}^m$. In order for multiplication by the complex exponentials (9) to take $\mathcal{H}^2(T_B)$ to itself for all $\eta \in \mathbf{R}^m$, it is necessary and sufficient for $B$ to be bounded, which ensures that the complex exponentials are bounded functions on $T_B$. On each slice of the form $\mathbf{R}^m + i\,y$, the modulus of the complex exponential (9) is equal to $\exp(-2\pi y\cdot\eta)$, which is constant on the slice. Thus if one multiplies $F(z)$ by (9), then (10) is multiplied by $\exp(-4\pi y\cdot\eta)$. This is more complicated than a translation in the $\mathbf{R}^m$ direction, since this factor depends on $y$ as well as on $\eta$.

Elements of the Hardy space have a nice Fourier-transform representation. Namely, $F(z)$ lies in $\mathcal{H}^2(T_B)$ if and only if $F(z)$ can be expressed as

$$F(z) = \int_{\mathbf{R}^m} \exp\Big(2\pi i \sum_{j=1}^{m} z_j\,u_j\Big) f(u)\,du, \quad z\in T_B, \tag{11}$$

where $f(u)$ is a measurable function on $\mathbf{R}^m$ such that

$$\sup_{y\in B}\int_{\mathbf{R}^m} |f(u)|^2\,\exp(-4\pi y\cdot u)\,du < \infty. \tag{12}$$

Furthermore, $\|F\|^2_{\mathcal{H}^2(T_B)}$ is equal to the quantity in (12) in this case. See [21, p. 93]. As usual, translations of $F(z)$ correspond to multiplications of $f(u)$ by exponentials, and multiplications of $F(z)$ by complex exponentials (9) correspond to translations of $f(u)$.

It is a classical result that if $B$ is a connected open subset of $\mathbf{R}^m$ and $\widehat{B}$ denotes the convex hull of $B$, then every holomorphic function on $T_B$ admits a holomorphic extension to $T_{\widehat{B}}$. See Theorem 9 in Chapter V of [4]. In fact $B$ is convex if and only if $T_B$ is a domain of holomorphy, which basically means that there is not some universal holomorphic extension of holomorphic functions on $T_B$ to a connected region that intersects $T_B$ but is not contained in $T_B$. See Theorem 3.5.1 in [15], for instance. For an element of $\mathcal{H}^2(T_B)$, one can also get this holomorphic extension from the preceding Fourier-transform representation, and, moreover, the Hardy-space norm of the extension is equal to the Hardy-space norm of the original function on the original domain. This is Corollary 2.4 on p. 93 of [21]. Note that for a nonempty convex open subset $B$ of $\mathbf{R}^m$, $\mathcal{H}^2(T_B)$ contains a function which is not identically 0 if and only if $B$ does not contain a line, as in Corollary 2.6 on p. 94 of [21].

Let us now consider the case of *convex cones*. Recall that a subset $\Gamma$ of $\mathbf{R}^m$ is said to be a convex cone if $t\,v \in \Gamma$ whenever $v\in\Gamma$ and $t$ is a positive real number, and $v + w \in \Gamma$ whenever $v, w \in \Gamma$. We shall assume as well that $\Gamma$ is nonempty and an open subset of $\mathbf{R}^m$ and that it does not contain 0, since otherwise $\Gamma$ would be all of $\mathbf{R}^m$. The *dual* $\Gamma^*$ of $\Gamma$ is defined by

$$\Gamma^* = \{u \in \mathbf{R}^m : v\cdot u \geq 0 \text{ for all } v \in \Gamma\}.$$

It is easy to see that $\Gamma^*$ is always a convex cone that is a closed subset of $\mathbf{R}^m$.

Note that $\Gamma^*$ is equal to the set of $u$ in $\mathbf{R}^m$ such that $\exp(2\pi i z\cdot u)$ is a bounded function of $z$ on the tube $T_\Gamma$ over $\Gamma$, and indeed

$$|\exp 2\pi i z\cdot u| \leq 1$$

for all $u \in \Gamma^*$ and $z$ in $T_\Gamma$. Now consider the Hardy space $\mathcal{H}^2(T_\Gamma)$. Because $\Gamma$ is unbounded, multiplication by the complex exponential (9) does not send $\mathcal{H}^2(T_\Gamma)$ to itself for arbitrary $\eta$ in $\mathbf{R}^m$, but this is true when $\eta$ is an element of the dual cone $\Gamma^*$. More precisely, if $F(z)$ is an element of $\mathcal{H}^2(T_\Gamma)$ and $\eta$ lies in $\Gamma^*$, then the Hardy-space norm of $F(z)$ times the exponential function (9) is less than or equal to the Hardy-space norm of $F(z)$ itself.

For a tube over a convex cone, the Fourier-transform representation of elements of the Hardy space can be converted into the following. A function $F(z)$ lies in $\mathcal{H}^2(T_\Gamma)$ if and only if it can be expressed as

$$F(z) = \int_{\Gamma^*} \exp(2\pi i z\cdot u) f(u)\,du,$$

where $f(u)$ is a measurable function on $\Gamma^*$ such that

$$(13) \qquad \int_{\Gamma^*} |f(u)|^2 \, du < \infty.$$

In this case $\|F\|^2_{\mathcal{H}^2(T_\Gamma)}$ is equal to the quantity in (13). This is Theorem 3.1 on p. 101 of [21]. Let us emphasize that the integrals over $\Gamma^*$ are taken with respect to $m$-dimensional volume measure. If $\Gamma$ contains a line so that the closure of $\Gamma$ contains a line, through the origin, then $\Gamma^*$ is contained in a subspace of $\mathbf{R}^m$ of dimension strictly less than $m$ and the integrals over $\Gamma^*$ are automatically equal to 0.

A tube over a cone $T_\Gamma$ is invariant not only under the translations $z \mapsto z - \xi$, $\xi \in \mathbf{R}^m$, but also under the dilation mappings $z \mapsto r\,z$, where $r$ is a positive real number. If $F(z)$ lies in $\mathcal{H}^2(T_\Gamma)$, then $F(r\,z)$ also lies in $\mathcal{H}^2(T_\Gamma)$, and the norm of $F(r\,z)$ is equal to $r^{-m/2}$ times the norm of $F(z)$. More generally, if $\phi$ is an invertible linear mapping from $\mathbf{R}^m$ onto itself such that $\phi(\Gamma) = \Gamma$, then the extension $\Phi$ of $\phi$ to a complex-linear mapping on $\mathbf{C}^m$ defines a biholomorphic automorphism of $T_\Gamma$. Composition with these mappings $\Phi$ defines linear operators on $\mathcal{H}^2(T_\Gamma)$, in which the norm is multiplied by the square root of the reciprocal of the absolute value of the determinant of $\phi$. For some special cones $\Gamma$, there may be quite a lot of these linear symmetries.

A basic example is the cone consisting of the elements $v$ of $\mathbf{R}^m$ such that each component $v_j$ of $v$ is a positive real number. A linear mapping on $\mathbf{R}^m$ takes this cone to itself if and only if the matrix of the linear mapping in the standard basis has nonnegative entries and each row in the matrix contains at least one positive entry. In particular, this cone is *homogeneous*, which is to say that for every two elements $v$ and $w$ in the cone there is an invertible linear mapping on $\mathbf{R}^m$ which takes the cone to itself and which takes $v$ to $w$. Note that the dual of this cone consists of the elements $u$ of $\mathbf{R}^m$ such that each component $u_j$ of $u$ is a nonnegative real number.

The domain $\mathcal{U}_{n+1}$ defined in (6) is not a tube domain but nonetheless enjoys remarkable structure which is somewhat similar. This includes a large family of biholomorphic automorphisms given by affine mappings, namely those generated by the $A_{(z,t)}$'s defined in (7) and the nonisotropic dilations $\Delta_r$ defined in (8). The Hardy space $\mathcal{H}^2(\mathcal{U}_{n+1})$ associated to $\mathcal{U}_{n+1}$ is defined as the space of holomorphic functions $F(w, \sigma)$ on $\mathcal{U}_{n+1}$ such that

$$(14) \qquad \int_{\mathbf{R}} \int_{\mathbf{C}^n} |F(w, r + i|w|^2 + is)|^2 \, dw \, dr$$

is finite for every positive real number $s$ and uniformly bounded. The norm $\|F\|_{\mathcal{H}^2(\mathcal{U}_{n+1})}$ is defined to be the square root of the supremum of (14) over $s > 0$. Compare with [20, p. 532]. For any holomorphic function on $\mathcal{U}_{n+1}$, one can compose with an affine mapping $A_{(z,t)}$ or a dilation $\Delta_r$ and get another holomorphic function on $\mathcal{U}_{n+1}$. If the initial function lies in the Hardy space $\mathcal{H}^2(\mathcal{U}_{n+1})$, then one can check that the same is true after composition. Composition with the $A_{(z,t)}$'s does not change the Hardy-space norm, while composition with the dilations $\Delta_r$ changes the norm by a scale factor. In other words, the Heisenberg group again acts on the Hardy space, but this time one uses only compositions and not multiplications by exponentials. Concerning these and related matters, see [15], [19], [20].

In short, several complex variables, classical Fourier analysis, and Heisenberg groups are interweaved in a number of ways, with various domains, spaces of functions, and actions on them.

## Some Geometric Aspects

As on any Lie group, one can define left-invariant smooth Riemannian metrics on the Heisenberg group by choosing any inner product on the tangent space at the identity element and then extending that to the whole group through left translations. In other words, the inner product on the tangent space at a point $p$ is determined by using a left translation to move $p$ to the identity element and using the differential of this translation mapping to transport tangent vectors at $p$ to tangent vectors at the identity, where the inner product has been specified.

On the Heisenberg groups, as well as other nilpotent Lie groups, one has the extra ingredient of *dilations*. On any Riemannian manifold one can scale the metric simply by multiplying it by a positive real number, but with the dilations one can make a different kind of scaling, using the dilations as a change of variables.

On $\mathbf{R}^n$ with the standard Euclidean metric, these two kinds of scaling have the same effect on the metric. This is not the case on the Heisenberg groups or other nonabelian nilpotent Lie groups. Let us briefly describe two very interesting aspects of this.

As mentioned earlier, one can take the quotient of $H_n(\mathbf{R})$ by the subgroup $H_n(\mathbf{Z})$ to get a smooth compact manifold. Since we are considering left-invariant metrics on $H_n(\mathbf{R})$, let us agree to take the quotient of $H_n(\mathbf{R})$ by the action of the subgroup $H_n(\mathbf{Z})$ on the left as well. This permits us to get Riemannian metrics on the quotient from left-invariant metrics on $H_n(\mathbf{R})$. Using the dilations, one can show that the quotient space admits Riemannian metrics such that the maximum of the absolute values of the sectional curvatures is bounded by 1 at the same time that the diameter is as small as one likes. This is equivalent to saying that there are metrics such that the diameter is bounded by 1 and the maximum of the absolute values of the sectional curvatures is as small as one likes. For more about this kind of phenomenon, see [6], [7], [12].

In addition to left-invariant Riemannian metrics, one can consider sub-Riemannian geometry. For this one specifies a family of subspaces of the tangent spaces to a given manifold and measures distances between points by taking lengths of curves where the curves are restricted to have their tangent vectors in the specified subspaces of the tangent spaces. Of course the family of subspaces of the tangent spaces has to satisfy certain properties in order for there to be such a curve between any two points. In the case of the Heisenberg group $H_n(\mathbf{R})$, such a family of subspaces of the tangent spaces can be obtained as follows. At the identity element $(0,0,0)$ the tangent space can be identified with the vector space $\mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}$ in a natural way, and for the subspace one takes $\mathbf{R}^n \times \mathbf{R}^n \times \{0\}$. At other points one gets a subspace of the tangent space by making a left translation to reduce to this subspace of the tangent space at the identity element. This leads to a family of subspaces of the tangent spaces which satisfies the condition just mentioned, and it is also invariant under left translations as well as the dilations on the Heisenberg group.

In the context of the domain $\mathcal{U}_{n+1}$ in $\mathbf{C}^{n+1}$ defined in (6), one can look at this in a slightly different way. Namely, in the boundary $\partial\mathcal{U}_{n+1}$, one can choose the family of subspaces of the tangent spaces to be the maximal complex subspaces. These are preserved by the mappings $A_{(z,t)}$ and the dilations $\Delta_r$ discussed before, because these mappings take $\partial\mathcal{U}_{n+1}$ to itself and they preserve the complex structure.

Two very good references concerning sub-Riemannian geometry in general are [2] and [17]. One might be surprised at the extent to which familiar and basic topics from advanced calculus become quite tricky in the sub-Riemannian case and for which there are a lot of open questions. In this regard, see [16].

## References


[1] R. Beals and P. Greiner, Calculus on Heisenberg manifolds, *Ann. Math. Stud.* **119**, Princeton University Press, 1988.

[2] A. Bellaïche and J.-J. Risler, eds., *Sub-Riemannian Geometry*, Birkhäuser, 1996.

[3] C. Berenstein, D.-C. Chang, and J. Tie, *Laguerre Calculus and Its Applications on the Heisenberg Group*, Amer. Math. Soc. and Internat. Press, 2001.

[4] S. Bochner and W. Martin, *Several Complex Variables*, Princeton University Press, 1948.

[5] A. Brown and C. Pearcy, Structure of commutators of operators, *Ann. of Math.* (2) **82** (1965), 112–127.

[6] P. Buser and H. Karcher, Gromov's almost flat manifolds, *Astérisque* **81** (1981).

[7] J. Cheeger, K. Fukaya, and M. Gromov, Nilpotent structures and invariant metrics on collapsed manifolds, *J. Amer. Math. Soc.* **5** (1992), 327–372.

[8] L. Ehrenpreis, *Fourier Analysis in Several Complex Variables*, Wiley, 1970.

[9] J. Faraut and A. Korányi, *Analysis on Symmetric Cones*, Oxford University Press, 1994.

[10] J. Faraut, S. Kaneyuki, A. Korányi, Q. Lu, and G. Roos, *Analysis and Geometry on Complex Homogeneous Domains*, Birkhäuser, 2000.

[11] G. Folland, *Harmonic Analysis in Phase Space*, Ann. of Math. Stud., vol. 122, Princeton University Press, 1989.

[12] M. Gromov, Almost flat manifolds, *J. Differential Geom.* **13** (1978), 231–241.

[13] R. Howe, On the role of the Heisenberg group in harmonic analysis, *Bull. Amer. Math. Soc. (N.S.)* **3** (1980), 821–843.

[14] A. Korányi and H. Reimann, Foundations for the theory of quasiconformal mappings on the Heisenberg group, *Adv. Math.* **111** (1995), 1–87.

[15] S. Krantz, *Function Theory of Several Complex Variables*, second ed., AMS Chelsea Publishing, 2001.

[16] V. Magnani, *Elements of Geometric Measure Theory on Sub-Riemannian Groups*, thesis, Scuola Normale Superiore, Pisa, 2002.

[17] R. Montgomery, *A Tour of Subriemannian Geometries, Their Geodesics and Applications*, Math. Surveys Monogr., vol. 91, Amer. Math. Soc., 2002.

[18] W. Rudin, *Functional Analysis*, second ed., McGraw-Hill, 1991.

[19] ——, *Function Theory in the Unit Ball of $\mathbf{C}^n$*, Springer-Verlag, 1980.

[20] E. Stein, *Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton University Press, 1993.

[21] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton University Press, 1971.

[22] M. Taylor, *Noncommutative Harmonic Analysis*, Math. Surveys Monogr., vol. 22, Amer. Math. Soc., 1986.

[23] W. Thurston, *Three-Dimensional Geometry and Topology*, S. Levy, ed., Princeton University Press, 1997.

[24] S. Vági, Harmonic analysis on Cartan and Siegel domains, in *Studies in Harmonic Analysis*, MAA Stud. Math., vol. 13, Math. Assoc. America, 1976, pp. 257–309.

[25] E. Vesentini, Invariant metrics on convex cones, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **3** (1976), 671–696.