Teaching Mathematics in the United States

Al Cuoco

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The author wrote the present article for the *Bollettino* in order to give Italian readers background about the U.S. system of mathematics education. Many U.S. mathematicians themselves have little contact with the public schools. Therefore this article, originally written for a foreign audience, is reprinted here in the expectation that it will also be enlightening to U.S. readers.

—Harold P. Boas

This is a companion to my article *Mathematics for Teaching.*[1] In it I'll set the context for the complex situation in the United States and go on to describe some of the ways in which teachers here are prepared.

One of the most striking features about the U.S. education system at every level is its decentralization. Precollege education (grades kindergarten through 12) is publicly supported through taxes and is compulsory through age sixteen. But there is no national curriculum. Each of the fifty states is free to design its own program, and, while there are state "curriculum frameworks", in most of these states, that freedom is passed on to each city and town. There are several "adoption states" (California and Texas are the largest) in which there are statewide lists of admissible texts, but they are exceptions; curriculum is largely a purely local choice. In fact, almost everything about schooling seems to be determined locally. For example, each town's teachers have their own union (affiliated with state and national organizations), and each local union negotiates the salaries of its members directly with its town government.

This decentralization is due to the way precollege education is funded. Although the federal government does funnel money to states, most funding for education is raised at the state and local level. My state (Massachusetts) is one of the most extreme in this regard—most public education is funded by taxes on one's property that are paid to the town government. This gives rise to huge differences in education spending, with wealthier towns having much better facilities, teacher salaries, and services. Using Massachusetts as an example again, per-pupil per-annum expenditures range across the cities and towns from about \$6,000 to \$12,000 if one ignores outliers [16]. Not surprisingly, these figures correlate well with the percentage of graduates who enroll in college—from less than 50 percent to over 95 percent.

The system of undergraduate education is even more complex. For the most part, admission to four-year programs is competitive, and students have to apply to individual institutions. There are private colleges and universities, supported completely by tuition, grants, and private endowments. And there is a parallel system of state colleges and universities, partially supported by tax dollars. All these institutions charge tuition, ranging from about \$10,000 per year (including room and board) for state colleges to something close to \$35,000 for private universities. These tuitions impose severe financial hardships on students and their families. In many cases where a family has more than one

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[1] Notices ***48*** *(2001), 168–174.*

college-age student, tuition and fees exceed the family's annual income. As a result, there is a vast array of loan programs and scholarships available for needy students. It is not uncommon for a student to graduate facing a $50,000 debt for college loans.

In addition to four-year schools, there is a system of two-year colleges. These usually have open admission, are publicly supported, and attract a wide array of students: students who want preparation for a profession that does not require a four-year degree, students who need some developmental work before transferring to a four-year program, and students who simply cannot afford the expenses of a four-year college. Since most two-year colleges are not residential, tuition is per course, and that averages $300 for a one-semester three-hour course. Many students who attend these schools also hold full-time jobs (more than a few support families), so even this tuition poses a hardship. An estimated 40 percent of teachers take some of their mathematics preparation at two-year schools [24].

Given this organizational patchwork, there is a surprising uniformity in American education. National forces, some of them having little to do with educational principles, tend to smooth out the irregularities of local control.

At the precollege level, one of these forces is the economics of textbook publishing. Publishers invest heavily in meeting the criteria of Texas, California, and other adoption states. As a result, the tables of contents in most major texts closely resemble the union of the curriculum frameworks in these key states. Because state curriculum frameworks have become essentially lists of specific and low-level topics to be covered (e.g., "combining fractions with unlike denominators"), mainstream commercial texts are a collection of loosely related chapters each treating one such topic. As one moves up the grades, the effects of this topic-driven design principle compound. By the time one reaches the fourth year of high school, we find 18-chapter, 700-page compendia of topics that range from triangle trigonometry to data analysis to complex numbers. The de facto national curriculum therefore consists of locally chosen subsets of topics from these texts.

Economic and social forces have, in other ways, always played an important role in public precollege education in this country [1]. On the one hand, working-class parents see schooling for their children as "the great equalizer" and education as a ticket to upward mobility in American society. On the other hand, there are subtle but substantial forces on schools to act as enculturation mechanisms (building "good citizenship"), to maintain economic inequity among various classes, and to produce graduates who fit into a work force that is based on hierarchy. This tension plays out in American classrooms every day: on the one hand, teachers work very hard to help their students become creative thinkers and problem solvers and to build their skills and their self-confidence. On the other hand, a great deal of time is spent ensuring that students learn to submit to authority and to obey an elaborate system of school rules that governs everything from punctuality to personal attire to when one can go to the lavatory.

An extremely potent smoothing force has taken root over the past decade. It is a phenomenon that might be more common in other countries than it has previously been here: the high-stakes exam. Taxpayers, through elected officials, are demanding accountability on the part of what they see as autonomous school departments. So all over the country students are required to pass certain exams before they can move on to the next level of schooling. Typically, there are three such exams in each discipline. In Massachusetts the mathematics exams are given at the fourth, eighth, and tenth grades. Students who do not pass the tenth-grade exam will not graduate from high school until they do pass it. What may be unusual in the U.S. system is that, true to the spirit of local control, every state makes up its own set of "standards" (usually lists of topics that will be tested at each grade; see [2] for a critique of these documents) and designs its own exams. Proponents of various particular educational philosophies are therefore lobbying in different states to gain influence over those who are in control of the standards and exams. Teachers are under enormous pressure to prepare their students to pass these exams. Students who fail the exams are put into "test-prep" classes that concentrate on the fine points of taking tests. Scores are published in the newspapers, schools are judged by the percentage of students who pass the tests, and the scores of a town's school system have an impact on the real estate values in that town: property values go up in districts with high scores (so they end up with even more resources to spend on education), and underperforming schools experience a drop in real estate values and hence a decline in the tax revenue that can be used to support education.[2]

Finally, many professional societies are filling the void left by the absence of an official national curriculum. In 1989, in response to a growing dissatisfaction among mathematics teachers in the U.S., the National Council of Teachers of Mathematics

[2] *Added in May 2003. New federal legislation (the "No Child Left Behind Act") has strengthened both traditions of local control and high-stakes accountability: more of the federal education budget is being distributed to individual states, and in return states are being required to greatly increase their testing programs and to provide (quantitative) evidence of success for funded programs.*

(NCTM) published its *Curriculum and Evaluation Standards* [20]. While not a curriculum, this publication called for more student-centered, activity-based programs for children and a diminished emphasis on rote memorization and technical drill. The *Standards* spurred an enormous amount of activity in curriculum development, in teacher preparation, and in professional development programs for practicing teachers, and it became the model for state frameworks and standards. The National Science Foundation (NSF) invested heavily in innovative curricula that gave specificity to the NCTM *Standards*. The *Standards* also spurred a substantial backlash to some of the more extreme interpretations of the report; more about this in the section on politics. A recent and extensive revision [21] attempts to fine-tune some of the recommendations and address some of the excesses that were carried out in the name of the original.

The American Mathematical Association of Two-Year Colleges produced a similar set of guidelines [3], and the Conference Board of the Mathematical Sciences, an umbrella organization for sixteen professional societies, has just published a set of recommendations [4] for the mathematical preparation of teachers. We will look more carefully at that in the section on teacher preparation.

Even though there is no national curriculum, federal agencies, especially the National Science Foundation and the U.S. Department of Education, are heavily involved in improving mathematics education across the country. NSF supports the development of curricula, professional development programs for practicing teachers, and innovations at the teacher-preparation level, and it is especially interested in increasing the mathematical competence of the teaching profession. Both agencies fund research aimed at finding more effective teaching methods and at obtaining better understanding of how young students come to understand mathematics.

So, overlaid on the collage of local decision making and control is a web of forces that tend to blur distinctions and to make the *effect* of education fairly uniform. That is why, for example, in *Mathematics for Teaching* I can point to what seem to be pervasive phenomena in the mathematics classrooms of teachers who, on the surface, seem to have had preparations that are quite varied. In the following sections I will elaborate a bit more on these similarities, especially as they pertain to mathematics teaching and teacher preparation.

It goes without saying that what follows is just the perspective of one person, clouded by a rather idiosyncratic (and therefore, by the above remarks, rather typical) indoctrination into mathematics education. Other perspectives abound (see [9], for example).

The Profession

The structures and forces described in the previous section tend to shape the teaching profession in several ways. In this section I will concentrate on teaching at the high school (grades 9–12) level, because that is the level with which I am most familiar.

The tradition of local funding for education has a direct impact on teachers' working conditions and salaries. With notable exceptions (in wealthy suburbs), starting mathematics teachers can expect to earn 50–75 percent of the starting salaries for the other professions that attract mathematics majors. Salary increments are a function of both years on the job and postgraduate courses. These postgraduate increments are for any kind of course taken, and a whole industry has grown up that provides teachers with convenient workshops, day-long seminars, or weekend courses that advance them on the salary scale. These courses vary wildly in quality and intensity, and they are often devoted to teaching techniques, use of technology, or classroom management. The professional development courses in mathematics that I describe in *Mathematics for Teaching* are among the rare exceptions to this statement.

> *Most public education is funded by taxes on one's property that are paid to the town government. This gives rise to huge differences in education spending.*

In Massachusetts yearly salary increments are negotiated between the local unions and the local school boards, so they have to compete with similar negotiations with other unions and with the local operating budget and tax base, both of which are constrained by law (in Massachusetts a town's budget cannot increase more than 2.5 percent per year without a very-difficult-to-obtain "override"). It is not uncommon for negotiations to produce no raise at all for several years in a row.

A typical school day starts between 7:00 and 8:00 a.m. and runs through midafternoon. Teachers teach four to six classes each day, usually in three or four different courses. Classes last almost an hour. In addition to their teaching duties, teachers usually spend one class a day "supervising" a study hall or a lunchroom and are given one "free" period to plan lessons and to grade papers. In most districts teachers are required to remain in the school for the entire school day, even during their free periods. Typical class size is 25–35 students.

Many teachers have developed substantial expertise in the use of technology. There was a period in the 1980s when schools invested heavily in microcomputers, but the educational use of technology has become almost completely confined to the use of calculators: numerical calculators in early grades and scientific (graphing) calculators in later grades. A small percentage of teachers uses dynamic geometry environments, spreadsheets, statistical packages, and even computer algebra systems (CAS). But until these media are available on handheld devices (as several now are), their use in education will remain confined to a small number of enthusiasts. Most uses of these computational environments are as replacements for paper-and-pencil calculations and, especially in the case of geometry, as a means for justifying conjectures. This poses a quandary for many teachers who question the usefulness of many topics they teach in light of the capabilities of mathematical software. This is especially true for the use of CAS technology. I have heard many teachers worry out loud that the existence of CAS environments on handheld devices makes a good deal of the current algebra curriculum obsolete. It is no coincidence that CAS technology has been slow to take root in high schools.

Although some schools use an "integrated" curriculum, the typical high school program still follows the American tradition of four courses: elementary algebra, geometry, advanced algebra, and "precalculus" (a mix of trigonometry, analytic geometry, and function graphing). Students who manage to start the sequence a year early can opt for a calculus course in their last year. Recent trends have infused some of the standard courses with statistics, probability, and combinatorics.

As uniform as this sounds, there is wide variation among schools and even within a school. The differences among schools can be related again to the financial resources communities are able and willing to devote to education. The differences within a school are another matter.

There is a widespread belief in the U.S. that the ability to succeed at mathematics is somehow an innate "all or nothing" affair: either you are destined to be a scientist, engineer, or mathematician or you will never be able to understand anything about mathematics. Young children who do not catch on to mathematics (typically arithmetic) at an early age are often deemed mathematically "slow" and are gradually moved into the group of students who "can't do math". By the time these students get to high school, many find themselves in the low tier of a tracking system that has as many as five different "ability levels" for each course. While the top few levels usually do a fine job preparing students for college (and even sometimes of giving them a glimpse of what mathematics is about), the lower levels are abysmal rehashes of elementary school low-level topics: the algebra is little more than drill in numerical and symbolic calculations, and the geometry is devoid of proof and consists mainly of vocabulary and practice applying area formulas. Many students in these tracks fail one of these courses and either have to repeat it or drop out of mathematics altogether.

Although students are placed in these low tracks for all kinds of reasons, children from dysfunctional families almost always end up here. These are precisely the students who feel most oppressed by the enculturation function of schools, and, with little support at home and no intellectual satisfaction from their courses, many act out and become difficult to control, a phenomenon that tends to spread outside the classroom. This is the underbelly of American education; it is a primary source of the disengagement from learning that affects too many adults, and it is a breeding ground for a great deal of the violence one reads about in American schools.

I am afraid I have painted a pretty bleak picture of the teaching situation in American high schools. In fact, low pay, oppressive workloads, rigid rules of behavior for students, and extensive sorting of students into tracks by no means exhaust the collection of challenges teachers face. I did not mention the low esteem in which much of the American public holds teachers (Mark Twain had a saying that is very popular in the U.S.: "Those who can, do; those who can't, teach"), nor did I discuss the fact that many teachers need to hold a second job (usually having nothing to do with education) in order to make ends meet. As I said in *Mathematics for Teaching*, while the mathematical preparation of teachers is essential to improving American mathematics education, it is in no way the only problem facing our schools, and in many ways it is not the most difficult one.

Nevertheless, there *is* an attraction to teaching for many of us. I know and work with hundreds of high school teachers, and the vast majority of them are extremely dedicated to their students and view their work as something very important. I have seen some pretty weak mathematics in classes I have visited, and I know a great many teachers who think that their primary job is to help students build "good" values and develop respect for authority, but I have seldom known a teacher who did not care about her students or work very hard to help them succeed.

And, oddly enough, some of the very things that make teaching in this country so frustrating contribute to the attraction of the profession. Let me cite one example from my own background. The administration in my school and the teachers in my department (in a working-class city outside Boston), in spite of wide differences in philosophies and values, were dedicated to the well-being and advancement of our students. By the time students in the bottom track of the system got to high school, there was little we could do to help them catch up so that they could take upper-level courses. But many of my colleagues and I found it extremely gratifying to work with these students. Because we had the trust of the administration and because, for the most part, we were not preparing these students for college, we could more or less do anything we wanted with them. So, many of the "low-level" courses at my high school turned into problem-solving experiences in which students designed and executed projects, often using the Logo computer language. It caused quite a shift in my approach to teaching when I realized that these students, the ones who "couldn't do math", were every bit as able to think in characteristically mathematical ways as students in my advanced courses. And although these students did not have the technical backgrounds necessary to advance in the usual curriculum, many did take more mathematics, and a few ended up in our advanced "independent study" elective. Furthermore, as my own mathematics education progressed, I began to see ways to circumvent the horrible texts that were dominant in the 1970s and 1980s, and I realized that while frontline research problems in mathematics are out of reach for most secondary students, many, if given the chance, are capable of understanding and using *methods* common among research mathematicians. This led to an approach to teaching and learning that kept me in the classroom for over twenty years and that has been the cornerstone of my subsequent work in education [7].

The Politics of Mathematics Education

If you have not been following the situation in the U.S., it may come as a surprise to hear that there is a furious debate here over the most effective ways to teach mathematics. Indeed, a Google search on "math wars" will turn up thousands of newspaper articles and websites arguing for or against this or that approach to teaching mathematics. Some of these involve interviews with eminent mathematicians or high government officials; others are written by parents, teachers, or business people. There have been television programs devoted to mathematics education, and on several occasions people have testified before Congress on the topic.

This math wars phenomenon is an extremely complex one; to do a good job of documenting and analyzing the events would take us too far afield (and such an account should be carried out by sociologists and anthropologists). As someone who feels part of several of the communities who are so angry with each other, let me give the briefest sketch of the landscape at a level of abstraction that will leave out many important details. The point that I want to make is that in the U.S., mathematicians and mathematics educators live in different worlds: they have different cultures, different standards of rigor, and even different languages for talking about mathematics and mathematics learning. At most universities, mathematics educators (people specializing in teacher preparation, epistemology, or curriculum design) are not part of the mathematics department; they belong to "schools of education", departments whose members are scholars in education first and disciplinary specialists second. And, at least for the past few years, significant numbers of people from these two cultures—mathematicians and mathematics educators—have been struggling to gain influence over each other and to wrest control of mathematics education in the U.S.

In the previous section I mentioned the poor curricula that were in place during the late 1970s and 1980s. These were in reaction to the excesses of the "new math" reform movement a decade earlier, a movement that was led by many prominent mathematicians and that tried to help children learn mathematics through deduction, logic, and mathematical structure. The emphasis in curriculum design was on logical precision, careful definitions, and polished presentations. In fact, there were some lasting benefits from the movement; the stylized caricatures of some of the most extreme aspects of the program mask the fact that there were some solid ideas here. But reaction to the curricula and philosophy grew into a movement that is sometimes called "back to basics", a slogan for an approach to mathematics that kept the topical organization and the vocabulary of the reform texts but eschewed abstraction and proof, emphasizing "basic skills". Far from basic, this approach evolved into a program full of arcane exercises that had little to do with basic skills or with mathematics. I can still find, in a very popular algebra book of that era, a page entitled

"Factoring $x^2 + bx + c; c$ Positive".

The page was filled with forty identical (and trivial) exercises. You can guess the title of the next page. Most high school classrooms followed a predictable format: the teacher would carefully work out an example of how to, say, add rational expressions. The students would try another similar example at their seats. The work would be checked, and then students would work on a practice set that continued on for homework that night.

By the mid 1980s the situation had gotten intolerable for teachers and students. Teachers (even those with weak mathematical preparation) were very uncomfortable with the technique-driven curriculum; students were dropping out of mathematics, mainly from sheer boredom, and even those who stuck with it, taking four years of high school mathematics, had no sense for what the discipline is about or what it is for. Advances in technology were making obsolete most of what was in the curriculum: numerical and symbolic calculations for the sole purpose of arriving at answers to pointless exercises.

At the same time, a critical mass of education researchers was adopting a neo-Piagetian philosophy of learning ("constructivism") that held that learning takes place when (and only when) learners build mathematical ideas in their own minds through a process of reflective abstraction. A corollary of this philosophy is that the kind of rote drill in computational technique that had become the staple of precollege mathematics would never produce robust mathematical understanding. This dovetailed perfectly with the years of anecdotal evidence built up in the teaching profession, where every teacher told stories of students who could imitate and execute all the routines but had no idea how to use them. It also flew in the face of the common sense of many mathematicians who believe that mathematics is best learned by first setting solid foundations and then advancing via precise explanations and ample practice.

In 1989, after considerable feedback from teachers in the field, the National Council of Teachers of Mathematics produced its *Standards* documents. I am leaving out a great deal of detail here; the NCTM leadership, largely a group of volunteers, had been working on policy documents for at least a decade, trying to reform precollege mathematics education into something that was more meaningful for students. Emerging from the NCTM deliberations was a vision of classroom organization that looked quite different from the classical lecture and recitation model. A notion was evolving among educators that perhaps one *should not* teach in the way one was taught. It was common during these times to look back with disdain at the experiment of the new math as overly pedantic and even as a stifling influence on children's mathematical development; people mocked the "Bourbaki influence" on precollege education. For these and other complex reasons, resentment among educators for mathematicians (especially those with no experience in precollege education) was growing.[3]

The timing of the release of the NCTM *Standards* was perfect. Teachers had just graduated a generation of students that was baffled by mathematics. Teachers saw this document—one that called for emphases on sense making, on looking at the utility of mathematics in other fields, on listening to and taking seriously students' ideas, and on reasoning and communication—as something that was speaking for *them*, telling people outside education that things had to change.

No one could have predicted what would happen over the next few years, and although hindsight provides some explanations, I still find much of it quite amazing. The different cultures of mathematics and mathematics education supported completely different interpretations of the same recommendations (and even sometimes of the same word). The "vertical disconnect" between the mathematics of the undergraduate and precollege curricula that I describe in *Mathematics for Teaching* contributed to some strange developments: given the license to downplay the silly treatment of topics in many texts and having never experienced, as part of undergraduate studies, the central position of school topics in the larger mathematical landscape, some teachers and educators proposed to stop children from memorizing multiplication facts, to eliminate the study of computational algorithms (some people even called them "dangerous"), to avoid the quadratic formula at all costs, to eliminate the study of conic sections, and to move algebra away from the study of formal calculations and toward the study of continuous variation. Of course, none of these was an explicit recommendation of the *Standards*, but the gates seemed to open for all kinds of recommendations, especially those advocating the abandonment of technical fluency and memorization, all in the name of reform.

As just one example, look at the role of proof in precollege mathematics. For mathematicians, the activity of constructing a proof is a research technique. In school mathematics, especially during the 1980s, deduction had nothing to do with discovery, insight, or experiment. Indeed, proof was taught and practiced almost exclusively in the yearlong geometry course as a *post-facto* ritual for establishing facts, most of which seemed obvious in the first place. In an attempt to help students construct stylized proofs of already established facts, texts and curricula encouraged the organization of statements and reasons in a two-column format (statements in one column, reasons in the other). This device had been used in the U.S. for decades, but it gradually evolved from a system for organizing one's work to a method for *constructing* proofs. This seems to be a recurring pattern in U.S. education (due again, in part, to the vertical disconnect and to the way university courses are organized): the way results and insights are *presented* becomes identical with how they allegedly are *conceived*. When "writing" a proof, many students in geometry would write the "given" on the top line of a two-column set-up (called a "T-bar

[3] *At the 1988 ICME in Budapest, I attended a long talk given by a prominent education researcher, who opened with "We're finally getting the arrogant mathematicians out of education."*

template" by many teachers), write what they were told to prove on the bottom statement line, put "Side-Angle-Side" or "Corresponding Parts of Congruent Triangles Are Congruent" beside it as a reason (usually one of these was right), and fill in the rest with random statements and reasons, hoping for the best.

So when the 1989 *Standards* called for reduced attention to the two-column proof and increased emphasis on developing the skills needed to write proofs and to record the results of deductive arguments in an understandable narrative form, many of us applauded. But within months I began hearing people in education claim that proof was an obsolete topic for school geometry, especially since geometry software experiments provided such convincing evidence. More than a few teachers were saying, "We don't do proofs anymore." Proof had already been eliminated from the low tracks of geometry; it was now about to disappear at every level.

This infuriated many mathematicians. And the story was the same with algebraic and numerical calculations, factoring to solve equations, plotting points by hand, establishing trigonometric identities, completing the square, and a host of other topics. The "reform movement", as it had become known, called for reduced attention to how these topics had been corrupted in school mathematics. Because these corruptions had *become* the topics in the minds of many (the vertical disconnect again), these statements were interpreted as calls to eliminate the topics themselves. And what was heard by those not familiar with schools and education, being completely oblivious to the existence of the corruptions, was a call for the gutting of a great deal of core mathematics from the precollege curriculum.

I have seldom known a teacher who did not care about her students or work very hard to help them succeed.

Several prominent mathematicians mobilized to stop what they saw as the demise of mathematics in K–12 education, and they did this with the tools that mathematicians know best: piercing arguments that tore apart the "reduced emphasis" recommendations by showing how the topics that were allegedly on the chopping block were prerequisites for further study in mathematics, science, and engineering. This piqued the attention of many parents, who wanted no part of programs that put their children at a disadvantage for getting into college or succeeding in advanced courses. There were also searing attacks on mathematics educators, pointing out mathematical errors and misunderstandings in published articles and poking fun at the language used by education researchers to describe their work. Educators replied in kind, mocking naïve public statements about education made by mathematicians.

This acrimony has existed for close to a decade, and, like other conflicts of this type, as it matures the public pronouncements get less strident and more conciliatory. The authors of the revised *Standards* [21] went to great lengths to include professional organizations such as the AMS and the MAA (Mathematical Association of America), as well as individual mathematicians, in the revision. Most mathematicians now realize that mathematical expertise is a necessary but not sufficient prerequisite for quality teaching and that teachers and educators have important expertise to offer the effort to improve mathematics education. Conversely, published reports on teaching and learning [4], [8], [14], [17], [18], [19] and guidelines for federally funded professional development programs for teachers place a great deal of emphasis on the importance of content knowledge for teachers. An especially hopeful development is a consensus document for K–8 mathematics [23], written by a team of mathematicians, educators, and teachers, that shows in a concrete way how the expertise distributed across the entire mathematics community can be synthesized in extremely effective ways.

But in private I still hear a great deal of distrust and dislike on all sides. I worry that the fight has become an end in itself. If I am right, the real casualties of the math wars will be another generation of students that will be subjected to the kind of mathematical nonsense that can only be fixed if the adults who know how to fix it stop arguing.

Teacher Preparation

The previous two sections describe some of the forces influencing U.S. precollege education. These same forces exert influence over the programs at universities that prepare teachers. In this section I will describe how some of these forces play out. Again, I will restrict myself to the preparation and professional development of high school teachers. For a description of and recommendations for teacher preparation programs in grades K–8, see [4], [13], [23], [24], [25].

Most states require that high school teachers meet certain formal certification requirements. True to the topic-driven curriculum, these requirements amount to a list of courses to be taken: the equivalent of an undergraduate major in mathematics (35–40 semester hours of mathematics courses) combined with several courses in education.

The mathematics courses are usually not designed specifically for prospective teachers. This is not necessarily a drawback: there are many benefits to studying mathematics for its own sake, and there is a great deal of effort among many undergraduate instructors (especially around the uses of technology) to make these generic courses appealing to a wide audience of students. And there *are* special courses for prospective teachers, usually in geometry or "capstone" courses that make connections among topics in the undergraduate curriculum or to high school mathematics. But because most of the mathematics courses taken by prospective teachers have to meet the needs of a wider clientele (including future mathematicians), discussions of teaching, learning, and the precollege curriculum hardly ever occur. Indeed, many college students majoring in mathematics do not decide to become high school teachers until late in their undergraduate careers. This often forces them into an extra year of college in which they take the necessary education courses to obtain state certification.

Teaching high school involves fielding questions, picking at germs of insight in students' ideas, and redirecting classroom discussion on the fly.

The split between mathematics departments and schools of education translates into a split in the mathematical preparation of teachers. Education courses are taken in education departments, separate from mathematics, but like the mathematics courses, many of these are generic courses—like adolescent psychology and the history of education—not aimed at prospective mathematics teachers. There is usually a "methods" course that concentrates specifically on methods for teaching high school mathematics. These courses often use the NCTM documents [21] as a basis for studying effective techniques for getting students involved in mathematical activities, techniques that are seldom used in the undergraduate mathematics courses themselves. There are some truly exceptional education courses that I know about in the Boston area, courses in problem solving or in the teaching of algebra and geometry, that are every bit as mathematical as courses offered in mathematics departments, but these are not common across the country.

So the short story is that prospective high school teachers are given a set of mathematics courses and a set of education courses. Putting the two together is essentially the job of the student, not the university.

Mathematicians and mathematics educators alike realize that this structure is not working. At AMS, MAA, and NCTM meetings all over the country, people are giving talks about the need to integrate mathematics and education, to connect undergraduate mathematics and school mathematics, and to make undergraduate teaching a model for what we want high school teaching to be. Evidence exists ([15], for example) that other countries are able to bridge these divides, but it remains to be seen if U.S. education will be able to overcome the traditions and hostilities that make progress along these lines difficult.

A good example of a thoughtful attempt to bring some coherence and purpose to the mathematical education of teachers is a recent report with exactly this name [4]. Known as the "MET report", it makes some recommendations that show an intimate knowledge of the problems in teacher preparation.[4] These include (see `http://www.maa.org/cbms` for the exact wording in [4]):

- Prospective teachers need to develop a deep understanding of the mathematics they will teach.
- Prospective high school teachers of mathematics should major in mathematics and in their last year take a two-semester course connecting their college mathematics courses with high school mathematics.
- Courses designed for prospective teachers should develop careful reasoning and mathematical "common sense".
- Mathematics courses for prospective teachers should develop the habits of mind of a mathematical thinker and demonstrate flexible, interactive styles of teaching.
- More mathematicians should consider becoming deeply involved in K–12 mathematics education.
- The mathematical education of teachers should be seen as a partnership between mathematics faculty and mathematics education faculty.
- There needs to be greater cooperation between two-year and four-year colleges in the mathematical education of teachers.
- There needs to be more collaboration between mathematics faculty and mathematics teachers.

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[4] *For two reactions to the report, see the October 2001 issue of the* Notices, *pages 985–91.*

- Teachers need the opportunity to develop their understanding of mathematics and its teaching throughout their careers.

The report gives many more insightful details of the difficulties facing reform in teacher preparation, and it takes seriously the structural problems faced by universities—the necessary inclusion of prospective teachers in courses designed to meet the needs of students preparing for other careers, for example. It offers concrete suggestions for meeting its recommendations. MET promises to be very influential in teacher preparation, setting the course for reform over the next few years.

Unfortunately, the specific content recommendations in MET are influenced by the topic-driven nature of U.S. curricula, and while MET gets the statement of the problem exactly right, I am afraid its blueprint for a solution, at least at the high school level, is lacking. For example, its recommendations around abstract algebra have to do with justifying the rules of elementary algebra. Yes, abstract algebra gives an axiomatic foundation for the algebraic transformations involved in precollege algebra, but it does so much *more* than that. Its major themes—decomposition, extension, and representation [12]—underlie and connect huge segments of the precollege curriculum. Algebra shows why polynomial algebra, one of the few universal objects students meet in school mathematics, occupies such a central role in formal calculation. Similarly, number theory does help one understand unique factorization, but, more importantly, major themes like reduction and localization give one a theoretical framework for bringing out the importance of many topics in elementary arithmetic. And Gauss's brilliant breakthrough theory of cyclotomy ties together more topics from school mathematics than almost any other theory in undergraduate mathematics. Number theory is also a basic tool in the *craft* of teaching, especially in the often neglected mathematical techniques of task design [6].

And so it goes. Extension by linearity is central to linear algebra and finds applications in everything from high school geometry to trigonometry, but is never mentioned in [4]. Nor is extension by continuity, completion, or other basic themes in analysis that underlie many topics in school mathematics. The deep applications of multilinear algebra to almost every topic in the secondary curriculum that involves geometric or algebraic symmetry (see [5], for example) would help teachers see genuine uses of determinants in a curriculum that makes almost no mention of them anymore.

This is not the carping of someone interested in trading one set of recommendations for another. Everything I mention above turned out to have frequent, almost weekly, utility in my high school teaching (with every level of course)—this was, for me, mathematics for teaching. And none of it was ever highlighted in my undergraduate courses as a theme worth considering. Some of these things *were* mentioned in my undergraduate courses, but in the same breath the discussion turned to a low-level pedantic proof of something like the fact that $0 \neq 1$ (this is the undergraduate version of the "flatness syndrome" that I describe in *Mathematics for Teaching*). It was not until graduate school that I realized that there are central themes in mathematics and, not coincidentally, that these themes are essential tools in the teaching of mathematics. At the right level of abstraction and in the right contexts, mathematical themes like these would be ideal organizers for courses for prospective teachers.

I find another aspect of [4] disappointing. The recommendation "Mathematics courses for prospective teachers should develop the habits of mind of a mathematical thinker …" is extremely important for many of the reasons I describe in *Mathematics for Teaching*, especially since teaching high school involves fielding questions, picking at germs of insight in students' ideas, and redirecting classroom discussion on the fly. But I am convinced that topic-driven survey courses of the kind most undergraduates take (mainstream linear algebra courses, for example) are not the vehicle for doing this. It is not that they are bad courses, but they are designed for another purpose—that of exposing students to an established mathematical theory. To meet this recommendation, students need an immersion experience in mathematics similar to the one I describe for practicing teachers (PROMYS) in *Mathematics for Teaching*. For all the reasons described there, a sustained immersion in a focused part of the discipline is one of the most valuable experiences a prospective teacher can have. It is a shame that this was not an explicit recommendation in [4].

So far I have described the "typical" path to teaching in a high school via a teacher preparation program in college. In fact, many teachers find their way into classrooms via other routes.

The economics of the job market have put in motion a pendulum that swings from an oversupply of teachers to a teacher shortage. I did not go through a teacher preparation program as an undergraduate, but when I finished college in 1969, the pendulum was in the shortage state, so I got a job with little difficulty. During the 1980s declining enrollments and property tax revolts across the country caused massive layoffs of teachers and other public employees, and all of a sudden there was an overabundance of teachers looking for jobs. Many teacher preparation programs came close to closing down, and mathematics majors and laid-off teachers saw much more opportunity in the high tech industry than in education. Schools went for years without taking on new staff. When I started teaching, I was the youngest person in my department. When I left twenty-four years later, I was still almost the youngest member. Teachers of my generation are now starting to retire, and enrollment is going up again. So we are now in a period of severe teacher shortages. Coupled with the downturn in high tech, many people with technical backgrounds are looking to fill the void in schools, and schools are taking them in in droves.

There is evidence [19] that as many as 50,000 inadequately prepared teachers enter the profession each year. By one report [11], 33 percent of the practicing mathematics teachers have neither a major nor a minor in undergraduate mathematics, and these teachers teach 26 percent of the country's mathematics students. To make matters worse, large numbers of qualified teachers leave the profession within five years.

People with many kinds of backgrounds are filling mathematics teaching openings. Some are engineers or scientists with significant mathematics backgrounds that are quite different from the typical undergraduate mathematics major. Some are teachers from other fields—science or computer science and sometimes history and elementary education—who take open mathematics positions looking for more job security. There are even out-of-work mathematicians, trained as researchers in pure or applied mathematics, looking for teaching positions in high schools.

Whereas these "through the back door" entries into the profession are more numerous now than a decade ago, there have always been people who come to teaching from outside teacher preparation programs. Although the efforts to reform undergraduate teacher preparation are crucial, they are invisible to these teachers who enter the profession via other routes. To compensate for this and for the shortcomings of current teacher preparation programs, local districts, states, and the federal government have had to invest heavily in ongoing professional development programs for practicing teachers.

Professional development has become big business in the U.S. Millions of dollars are spent on programs every year, and many large cities have special departments in the central administration devoted to funding and implementing professional development programs. Most states require participation in such programs in order to maintain certification, and many universities and school districts provide alternative certification programs to help people gain the qualifications they need to teach while holding teaching positions via "provisional" certification.

The needs of the teaching force vary so widely that most systems opt for an eclectic menu of professional development offerings ranging from after-school classes to one-day workshops to organized sequences of such experiences. The content varies widely, too, covering everything from cooperative learning techniques, the use of graphing calculators, and seminars on how to implement a particular curriculum, to what has come to be known as "make and take" workshops, where teachers spend an afternoon or a day working through activities that they can use directly with their students. Because the programs are either one-day seminars or a set of such seminars separated by many weeks, it is very difficult to do any significant mathematics in these programs. The clearly specious assumption (that in *Mathematics for Teaching* I call "know it all before you start teaching") is that the mathematics needed by teachers was learned in college.

Concluding Remarks

Mathematics as a scientific discipline is quite healthy in the U.S.; each year graduate schools produce a new corps of highly talented Ph.D.'s, many of whom join a research establishment that is among the most productive in the world. And, in addition to mathematical research, my country makes essential contributions to profound advances in technology, science, and finance. All these contributions rest on a bedrock of mathematical expertise that is as solid as any in the world. How then can an educational establishment that produces some of the best minds have so many weaknesses? One answer lies in the huge scale of the educational enterprise. Even if our mathematics programs lost half of our students for each of the twelve years of precollege education (as claimed in [22]), there would still be a large pool of young adults with the preparation needed to major in mathematics in undergraduate school. In fact, many argue that U.S. mathematics education has evolved into a system designed precisely to nurture, from the earliest grades, the talent that will eventually take leading roles in science and technology, often at the expense of a greater mathematical literacy for all high school graduates. Although I do not agree with this assessment, it is certainly true that the upper-level tracks in high school are often taught by the most mathematically expert teachers, and the curricula used in such programs are usually quite traditional, emphasizing the technical expertise needed to succeed in university majors in mathematics and science. And for the truly precocious students who show a knack for mathematics at a young age, there are many extra-curricular opportunities, from summer "math camps" to mentoring programs. The teacher counterpart for one such program (PROMYS) is described in *Mathematics for Teaching*. PROMYS (Program in Mathematics for Young Scientists) for *students* has been in existence for over ten years in Boston; it works with about sixty very advanced high school students each year, many of whom eventually specialize in mathematics or a related field. Similar programs exist at Ohio State and at other universities around the country.

So, preparation for students in the "top end" of the mathematics education spectrum seems to be working quite well. Indeed, the efforts of many of us to improve mathematics education for the rest of the spectrum can be thought of as an attempt to make the top end more inclusive, to awaken the nascent interest in mathematics that almost all students show when given a chance, and to prepare and develop mathematics teachers with the same success as that with which we prepare and develop mathematics researchers.

This sets a bit of the stage for my comments in *Mathematics for Teaching*. Public education in the U.S. is an extremely complex enterprise, and others in my country would have completely different perspectives about what would be important and interesting (see [9], [10], [26] for example). If I have conveyed a glimpse of the complexity of the system, then I have accomplished what I set out to do.

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