Laurent Schwartz (1915–2002)

François Treves, Gilles Pisier, and Marc Yor

Biographical Sketch

François Treves

Laurent Schwartz died in Paris on the 4th of July 2002. He was born in Paris on March 5, 1915. His father, Anselme Schwartz, had been born in 1872 in a small Alsatian town soon after the annexation of Alsatia by Germany. A fervent patriot, Anselme Schwartz had emigrated to France at the age of fourteen (before speaking French). In Paris he managed to carry out successful studies in medicine and was to become a prominent surgeon in France in the years between the two world wars. In 1907, having just become the first Jewish surgeon ever officially employed in a Paris hospital, Anselme married a first cousin, Claire née Debré. Anselme and Claire had been raised in the Jewish religion, but they brought up their three sons, Laurent, Bertrand, and Daniel, in very strict atheism. The extended Schwartz family was remarkably extended. Jacques Hadamard was Laurent's grand-uncle. An entire branch of the family were the Debrés: Laurent's maternal grandfather, Simon Debré, was the chief rabbi in Neuilly; later there was a Debré president of the national Academy of Medicine; there have been and there are prominent Gaullist politicians Debrés (by then Catholic converts). In 1938 Laurent Schwartz married the daughter of Paul Lévy, Marie-Hélène, who was to become a distinguished mathematician in her own right.

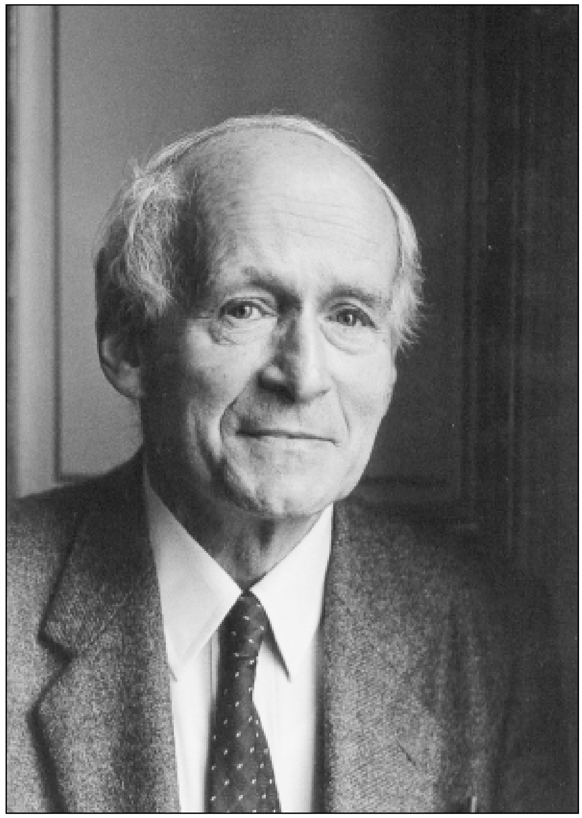
On completing high school (the French *lycée*), Laurent Schwartz hesitated between a career as a classics scholar and that of a mathematician. He had won the Concours Général in Latin; the Concours Général was and still is the most prestigious nationwide competition in France for high schoolers. Meanwhile he had become fascinated by the beauty of geometry, and, in the end, with the encouragement of one of his professors in classics and of his uncle, the pediatrician Robert Debré, and despite the rather unhelpful attitude of Hadamard, dismayed that the sixteen-year-old Laurent was not acquainted with the Riemann zeta function, he tried for admission to the science classes of the École Normale Supérieure (ENS), the most selective and most scholarly oriented of the "Grandes Écoles". He underwent the rather grueling two-year training ("hypotaupe", followed by "taupe", the tunnelling "submole" and "mole" years, so to speak) preparatory to entrance to the ENS, where he was admitted in 1934. Gustave Choquet, a winner of the Concours Général in Mathematics, had also passed the admissions exam that same year, and so had Marie-Hélène Lévy, one of the first *normaliennes*. Although the ranks of potential French scientists had been decimated by the First World War, there was still an impressive roster of mathematicians, saved from the carnage by their age, to whose teaching the young *normaliens* could be exposed: É. Borel, É. Cartan, A. Denjoy, M. Fréchet, G. Julia, P. Montel, to name some of the best known. At the nearby Collège de France they could also hear Lebesgue's lectures and take part in the Hadamard seminar. The enduring love of Laurent Schwartz for probability theory originated at that time, through his personal acquaintance and private conversations with his father-in-law-to-be, Paul Lévy.

In 1937, his studies at the ENS completed and the *agrégation* (the diploma needed to become a teacher in a lycée) secured, Schwartz let himself be drafted into military service, at that time compulsory in

François Treves is professor of mathematics at Rutgers University. His email address is `treves@math.rutgers.edu`.

France and, in principle, lasting three years. The idea was to get rid of the chore as early as possible and then to fully engage in his teaching and research career. During his years at the École Normale, Schwartz had become strongly politicized. His nascent political interest owed little to the influence of other students but much to his wide historical and political reading. It had been triggered by the revelation (gotten in some of the memoirs of the 1920s and 1930s) that the struggle in the First World War had not been the black-and-white affair painted in the French high bourgeoisie in which he had been raised. The Kaiser's Germany had not been quite the Evil Empire, but, yes, an empire and at the same time an imperfect and evolving democracy (with a substantial social-democratic party), not unlike France itself emerging from the Dreyfus affair. Another eye-opener for Schwartz was learning about the treatment of native peoples by the European colonial powers. The late 1930s were also the time of the Moscow trials; Schwartz perceived at once their true nature and meaning. He became and remained for the rest of his life an ardent anti-Stalinist (later, as a professor in Nancy he was accused in leaflets circulated by Communist-party students of being a CIA agent). It is not hard, more than sixty years later, to sympathize with that very idealistic, and moralistic, young Frenchman of the late 1930s, indignant at the evils of colonialism and opposed to the right-wing politicians whose hatred of the socialists was pushing them towards fascism, attracted to the far left (sole supporters of the soon-to-be-destroyed Spanish Republic) but repelled by Stalin's regime. With the Nazi threat looming ever larger, he became an active Trotskyite, and he remained one until 1947. It is striking, however, to realize how critical he was from the start of some of the political analyses and assessments by Trotsky and his followers. His political evolution brings to mind that of another French mathematician, Jean van Heijenoort, who had belonged to the same leftist circles as Schwartz, then abandoned mathematics to follow Trotsky as a kind of secretary from Istanbul to Mexico. Eventually he emigrated to the U.S. (enrolling as a Ph.D. student at NYU, where he shared an office with Louis Nirenberg) and in 1948 published in the *Partisan Review* a severe critique of the logical thinking in Marxism.

Politics, always viewed from the standpoint of a moralist, remained a permanent concern of Laurent Schwartz. After World War II he militated actively and very visibly against the French war and later the U.S. war in Vietnam (he was a member of the Russell tribunal) and most prominently against the French war in Algeria. A mathematics Ph.D. student at the University of Algiers, Maurice Audin, had been tortured and murdered by the French military. Schwartz chaired the "Comité Audin", demanding (unsuccessfully) an official enquiry into the murder. He also chaired the university committee that awarded a posthumous Doctorat ès Sciences to Audin. Jointly with Lipman Bers, Henri Cartan, Jean Dieudonné, and others, he agitated constantly to rescue mathematicians from repressive governments around the world: José Luis Massera in Uruguay, Jiri Müller in Czechoslovakia, Leonid Pliush in the USSR, to name a few. Defense of human (and not only mathematicians') rights took up much of his public activity, often in close collaboration with the classical historian Pierre Vidal-Naquet. He used to joke that one of the few merits he saw in being a member of the French Academy of Sciences was that the title might help impress some foreign official to "do the right thing". In the late 1970s he was among the relatively few European leftists who protested the Soviet invasion of Afghanistan.



Laurent Schwartz

Back in 1938, a recruit in the French army, Laurent saw clearly what had happened at Munich and that war was coming. War came, and it was no longer possible for the Schwartzes to live in northern France under German occupation. For a while Laurent continued to be paid a stipend by the office that eventually became the CNRS (Centre National de la Recherche Scientifique). The administrators seem not to have been paying much attention to the fact that he was Jewish. His employment, however, was abruptly terminated in 1942, after which (and until the end of the war) he received some modest financial support from the foundation Aide à la Recherche Scientifique, funded by Michelin, the tire production company, which did not abide by the racial policies of the Vichy government.

In 1940–41 the scientific environment in which Laurent and Marie-Hélène Schwartz found themselves was a bit of a desert. But luck helped them come across Henri Cartan and Jean Delsarte in Toulouse on a brief visit. Cartan and Delsarte, who were earlier acquaintances, informed them that many faculty members of the mathematics department of the University of Strasbourg (then the premier mathematics department in France) had moved to the University of Clermont-Ferrand. There they both could work under the guidance of some of the migrated mathematicians: J. Dieudonné, Ch. Ehresmann, A. Lichnerowicz, and S. Mandelbrojt, among others. The move to Clermont-Ferrand (the Schwartzes lived in very constrained circumstances in a nearby village, spending their weekdays at the university) represented an important turning point in Laurent's career. He met many of the members of the Bourbaki group and was introduced to their approach to mathematics. He became a "Bourbakist" as he had become a Trotskyite, deeply and lastingly influenced but far from uncritical. His major disagreement with Bourbaki lay far in the future. Interestingly, its roots were to be in the Bourbaki formulation of measure theory, exclusively concerned with Radon measures on locally compact spaces. Schwartz came to regard it as one of the major missteps of the group. Having himself been part of the writing committee, he was partly responsible. He was disturbed by the blatant inadequacy of the Bourbaki framework to accommodate the results of Paul Lévy and other probabilists, J. L. Doob among them, dealing with measures on infinite-dimensional spaces such as $C([0,1])$, definitely not locally compact. His renewed interest in probability theory later solidified his dissent.

The other main mathematical event in the Clermont-Ferrand period of Schwartz's life was the completion of his Ph.D. thesis on the approximation of continuous functions on $\mathbb{R}_+ = [0, +\infty)$ by sums of exponentials $S = a_0 + \sum_n a_n \exp(-\lambda_n x)$, where the infinite sequence of distinct real numbers $\lambda_n > 0$ is fixed. A theorem of Ch. Müntz implies that these sums are dense in $C(\mathbb{R}_+)$ if and only if $\sum_n \lambda_n^{-1} = +\infty$. Schwartz proved that when $\sum_n \lambda_n^{-1} < +\infty$, the closure in $C(\mathbb{R}_+)$ of the subspace made up of the sums S consists of the functions that can be extended holomorphically to the open half-plane $\Re z > 0$. His main tools were functional analysis and the Fourier and Laplace transforms, a kind of trial run for the work that was to make him famous.

In other times the doctorate would have opened the way to a straightforward career as a teacher in a university, first as a "Maître de Conférence", later as a full professor. But not much could be straightforward in such times. There was a rush of events, both on the world stage and at the personal level. In 1943 the Allied forces landed in North Africa, and the Germans invaded the south of France. The movements of a Jewish Trotskyite became very risky; false identities for the family had to be secured. In the middle of all this, in the hope of at least genetic survival, the Schwartzes decided to have a child. Marie-Hélène had a difficult pregnancy; the Vichy police raided the hospital where she had left her newborn son, Marc-André (on medical advice, so as not to risk transmitting to the baby the TB of which she seemed, but was not sure, to be cured). In the months that followed, the changes of address of the little family were unavoidable and frequent.

In the summer of 1944 Paris was liberated by the Allied forces, and soon thereafter the German troops were pushed back to the Rhine. In November 1944 (while still living under the assumed name of Sélimartin), Schwartz discovered distributions: as convolution operators $\varphi \to u * \varphi$ on the space C^∞_{comp} of test functions, not quite their final definition (that was to come to him in February 1945) as continuous linear functionals on C^∞_{comp}. After teaching at the University of Grenoble during the academic year 1944–45, he was invited to join the faculty in Nancy, where Delsarte and Dieudonné were hoping to create a world-class center of mathematics, a kind of mirror image of the Chicago mathematics department, where André Weil was at the time. Delsarte, Dieudonné, Weil, jointly with Henri Cartan, Claude Chevalley, and René de Possel, had been the "founders" of Bourbaki; and Nancago was the name of the fictitious "institution" that had started to bring the Bourbaki opus into print. The composition of the entity Bourbaki was ever-changing, upcoming young mathematicians replacing departing older ones. Schwartz was soon recruited into the group. In 1950 he was awarded the Fields Medal for his distribution theory. In 1952 he came to Paris as a professor at the Sorbonne.

In 1959 he left the Sorbonne to take up a professorship at the École Polytechnique (nicknamed L'X), where Paul Lévy had been teaching and where Schwartz embarked on an ambitious program of reform. Before Schwartz's appearance on the scene, the X did not produce any researcher in the sciences. It was and still is a military school, its president an army general. For ages it had been a top, if not the top, engineering school in France; its alumni were mostly engineers employed in industry, often in the high administrative echelons of private and, increasingly, public enterprises. Only a few students chose a military career. The entrance examination has always been very selective, more or less on a par with that of the École Normale Supérieure. In principle the students entering X were a terrific talent pool. Schwartz directed his considerable experience as a militant to convince officialdom and colleagues to help create the conditions that would attract young talent to research in mathematics as well as in related disciplines (especially physics). He recruited brilliant younger mathematicians to the "laboratoire de mathématiques", and he organized a seminar that became internationally known and is still functioning. One can safely say that his Polytechnique campaign was a resounding success. The school is now a very active research center, where some of the best mathematicians in France teach and do research and help students to start doing their own.

After Polytechnique, Laurent Schwartz fought a last campaign to make the universities more competitive and for what he called *la sélection*. He advocated injecting some selectivity into the rules of admission to the universities. The admission is open to anybody with an end-of-high-school diploma; the standards for this diploma have been declining. Historically speaking, his timing was poor. French academia was in rapid transition from a network of a few sparsely populated elite institutions (in 1952, circa 250,000 students in all subjects) to a mass education enterprise (close to 2.5 million students today). At the end of the twentieth century and still under the influence of May 1968, the dominant ideology has been that of equalitarianism. Any proposal with a hint of meritocracy has evoked suspicion and hostility. Even those colleagues who sympathized with his aims felt the fight was hopeless, too late, too incompatible with the need to educate large masses of youngsters not especially inclined towards scholarship. One can safely say that in the fight for sélection, his failure was total. As he himself used to say with a smile, who could have predicted that the Soviet regime would collapse before any change in the admission regime of French universities?

During his long career Laurent Schwartz had many students, and it will suffice here to name five of them, among the earlier ones and the most eminent: Louis Boutet de Monvel, Alexandre Grothendieck, Jacques-Louis Lions, Bernard Malgrange, and André Martineau.

No sketch of Laurent Schwartz's life can neglect to mention his achievements as a lepidopterist. Parts of his collection of tropical butterflies and moths, one of the greatest private collections in the world (with more than 25,000 specimens), have been donated to the Muséums d'Histoire Naturelle in Paris, Lyon, and Toulouse, and parts to a museum of natural history in Cochabamba (Bolivia). His collecting through the tropics led to the discovery and description of several new species now named after him.

No sketch of Laurent Schwartz's life can fail to recall his kindness. The intensity and range of his political commitment might give the impression of uncompromising radicalism. Actually he was extremely tolerant of other people's opinions. He confronted officials and officers with firmness but always with courtesy. Equanimity best describes his mental attitude, and the willingness to concede that he might be wrong.

Laurent Schwartz is survived by his wife, Marie-Hélène, and his daughter, Claudine Robert, a professor of statistics at that same University of Grenoble where he had held his first university position.

Distribution Theory and Functional Analysis

In the middle of the twentieth century, mathematics was ready for a satisfactory theory of generalized functions. The needs and the means to satisfy them were present.

The need was demonstrated by the various and recurrent attempts during the preceding half-century to define derivatives of functions that did not seem likely candidates for differentiation—step functions, for example. Often but not always these attempts were triggered by the need to solve differential equations, ordinary or partial. Two of the most significant (and successful) attempts had occurred at the beginning of the century: Heaviside's symbolic calculus, invented to solve the ordinary differential equations (ODEs) of electrical circuits, and Hadamard's *finite parts*, introduced to obtain explicit formulas for what are now called the fundamental solutions of the wave equation in higher space-dimensions. The "explanation" of Heaviside's calculus through the Laplace transform was not very convincing, but it pointed to some kind of link with Fourier analysis. Heaviside's calculus transformed convolution into multiplication, but the derivative rules for the convolution $(f * g)(t) = \int_0^t f(t-s)g(s)\,ds$ of two differentiable functions in the closed half-line $[0, +\infty)$,

$$\frac{d}{dt}(f * g)(t) = f(0)g(t) + (f' * g)(t),$$

were poorly understood. In 1926 Norbert Wiener used regularization, i.e., convolution with compactly supported C^∞ functions, to approximate continuous functions f by smooth ones.

An impressive widening of the kinds and uses of generalized functions came from theoretical physicists engaged in building quantum mechanics. On the real line the "Dirac function" was known to Heaviside: it was the function with symbol 1, the derivative of what we sometimes now call the Heaviside function, equal to 0 in $(-\infty, 0)$ and to 1 in

$(0, +\infty)$, whose symbol is $1/p$. Without making the link with Heaviside's work, Dirac used approximations of his "function" by true functions and introduced multidimensional Dirac measures, for instance, the measure associated to the light-cone $\Gamma \subset \mathbb{R}^4$,

$$\begin{aligned}\varphi \to \int \varphi(x)\delta(x_0^2 - x_1^2 - x_2^2 - x_3^2)\, dx_0 dx_1 dx_2 dx_3 \\ = \int_\Gamma \varphi(x) d\mu,\end{aligned}$$

where $d\mu$ is the natural invariant measure on Γ (in this equation φ can be any continuous function in $\mathbb{R}^4$ with sufficient decay at infinity). The late 1920s and the 1930s saw accelerated progress in this direction and in new ones:

Boundary-value problems for elliptic equations required handling of generalized functions more singular than Radon measures, e.g., multilayer boundary data.

In the last chapter of his book [Bochner, 1932], Bochner defines the Fourier transform of finite sums of the kind

$$\sum_{n=0}^{N} \frac{d^n}{dx^n}(P_n(x) f_n(x)), \tag{1}$$

where P_n is a polynomial and $f_n \in L^2(\mathbb{R}^n)$ ($N \in \mathbb{Z}_+$ can vary). This is essentially the definition of a tempered distribution. What is missing? Bochner looks at the derivatives d^n/dx^n as formal operations, not as weak derivatives; there is no duality, no space of (rapidly decaying) test-functions. The space of sums (1) is stable by (formal) Fourier transform. Fourier transform exchanges multiplication and convolution, although Bochner does not say when these operations are defined. He does not even point out that the Dirac function is of type (1). There is no evidence that he ever made anything of his generalized functions, not even when later (in 1946) he defined the generalized (truly, the distribution) solution of a linear ODE.

Schwartz did not know anything of Wiener's and Bochner's works, but he attended Leray's Peccot lectures at the Collège de France in 1934–35, in which Leray defined the *weak solution* of a second-order linear partial differential equation $P(x, \partial_x)u = 0$ in $\mathbb{R}^3$ by the property that

$$\int_{\mathbb{R}^3} u(x) P^\top(x, \partial_x)\varphi(x)\, dx = 0$$

for every compactly supported function φ of class C^2. Here $P^\top(x, \partial_x)$ denotes the transpose of $P(x, \partial_x)$; of course the coefficients of $P(x, \partial_x)$ must have a modicum of regularity, and u must be locally integrable. Leray did not define the derivatives $\partial_x^\alpha u$ for $|\alpha| > 2$.

The closest any mathematician of the 1930s ever came to the general definition of a distribution is Sobolev in his articles [Sobolev, 1936] and [Sobolev, 1938] (Leray used to refer to "distributions, invented by my friend Sobolev"). As a matter of fact, Sobolev truly defines the distributions of a given, but arbitrary, finite order m: as the *continuous linear functionals* on the space C^m_{comp} of compactly supported functions of class C^m. He keeps the integer m fixed; he never considers the intersection C^∞_{comp} of the spaces C^m_{comp} for all m. This is all the more surprising, since he proves that C^{m+1}_{comp} is dense in C^m_{comp} by the Wiener procedure of convolving functions $f \in C^m_{\text{comp}}$ with a sequence of functions belonging to C^∞_{comp}! In connection with this apparent blindness to the possible role of C^∞_{comp}, it is amusing that in 1944, when Schwartz mentioned to Henri Cartan his inclination to use the elements of C^∞_{comp} as test functions, Cartan tried to dissuade him: "They are too freakish (*trop monstrueuses*)."

Using transposition, Sobolev defines the multiplication of the functionals belonging to $(C^m_{\text{comp}})^*$ by the functions belonging to C^m and the differentiation of those functionals: d/dx maps $(C^m_{\text{comp}})^*$ into $(C^{m+1}_{\text{comp}})^*$. But again there is no mention of Dirac's $\delta(x)$ nor of convolution, and no link is made with the Fourier transform. He limits himself to applying his new approach to reformulating and solving the Cauchy problem for linear hyperbolic equations. And he does not try to build on his remarkable discoveries. Only after the war does he invent the Sobolev spaces H^m and then only for integers $m \geq 0$. Needless to say, Schwartz had not read Sobolev's articles, what with military service and a world war (and Western mathematicians' ignorance of the works of their Soviet colleagues). There is no doubt that knowing those articles would have spared him months of anxious uncertainty.

In the 1940s there were many different areas besides differential equations in which the need for universal concepts and tools was becoming clear. Three of them will be mentioned here.

Representation Theory for Lie Groups

The important book [Weil, 1940] of André Weil had already pointed to the relevance, in the absence of a differentiable structure on the group, of two of the most basic concepts of harmonic analysis: convolution and Fourier transform. (The definition by duality of a Radon measure in Weil's book should, but does not seem to, have been inspiring to Schwartz, who had read it.) In the 1950s and 1960s, F. Bruhat and Harish-Chandra were to show the role of distributions on a Lie group. The natural generalization of the Schwartz space S and of the Fourier transform to semisimple Lie groups was introduced by Harish-Chandra and put to spectacular use. Distributions provided a solid framework and language for noncommutative harmonic analysis.

Homology and Cohomology of Smooth Manifolds

Today we know that there is essentially one cohomology theory: cohomology with values in the appropriate sheaf. But in the 1940s there were various notions of homology and cohomology floating around. On a C^∞ manifold the theorems of De Rham, following up on Hodge Theory, pointed to duality between (singular) homology and (De Rham) cohomology, but the duality could not be formalized in mathematically acceptable analytic terms. On learning of distributions directly from Schwartz, De Rham saw at once the concept that had been missing: *currents*, which is to say, differential forms with distribution coefficients (see [De Rham, 1955] and [Schwartz, 1966]). The duality with (compactly supported) smooth differential forms is built into their definition, allowing the extension to currents of the exterior derivative and, on a Riemannian manifold, of the Hodge operations. Cocycles and coboundaries can then be conceived as analytic objects: closed and exact currents, respectively. Chains in singular homology, and thus very "concrete" geometric objects, are viewed as currents. Currents of weak regularity have turned out to be important also in the study of minimal surfaces and of analytic varieties.

Quantum Field Theory

As Schwartz liked to point out, in general physicists were better prepared to accept distributions than were mathematicians. Of course, physicists had invented some of them and had been using them for a while. They were accustomed to the idea that a functional could not, in general, be evaluated at a point, but had to be tested over an extended region. In fact, the axiomatic theory of quantum electrodynamics needs a highly sophisticated version of distributions: distributions with values in the set of unbounded linear operators on a Hilbert space (see [Streater, 1964]). There are serious difficulties associated with these objects, rooted in the impossibility of multiplying among themselves most *scalar* distributions. Since quantum electrodynamics is an incredibly accurate method for predicting experimental measurements, there must be ways of circumventing these difficulties. Therein lies in part the justification for renormalization.

As for the tools required to develop a unified theory of generalized functions, they were ready at hand in the 1940s; in fact, they were essentially available since the publication of the landmark monograph [Banach, 1932]. The Bourbakists were experts on the subject of the duality of very general locally convex topological vector spaces (a subject to which a crucial contribution was made by G. Mackey in the 1940s).

Thus there is no question that the times were ripe and that Laurent Schwartz was especially well placed to provide a theory of generalized functions. He said often that the invention of distributions would have occurred in any case, with or without him, and that it would have come within the next ten years at the latest. But in all likelihood the presentation would have been quite different from his own, heavily dependent on the theory of topological vector spaces.

A number of results intrinsic to distributions and partial differential equations cannot be proved without recourse to sophisticated functional analysis. But, undoubtedly, depending on it can be a hindrance to a quick introduction to distributions. Fortunately, in teaching it can easily be, and most of the time is, bypassed. Most of the basic tenets of the theory can be stated and proved using solely *sequences* of test functions or of distributions. The great success and usefulness of distribution theory lies in its simplicity and in the easy, automatic nature of its operations. Many have accused it of being "shallow". But that is precisely what analysis needed, a shallow justification for what it wanted to do: for instance, to differentiate under the integral sign without thinking twice. With the easy part taken care of, analysts could push further and take care of the finer and more difficult points.

Laurent Schwartz chose to spend half of the decade of the 1960s proving theorems about highly complicated topologies on tensor products of topological vector spaces to be able to study distributions valued in those spaces. Meanwhile, the tendency in large sectors of analysis was pointing in the opposite direction. In the past thirty to forty years the tendency has been to use scales of Banach spaces (such as the Sobolev spaces $W^{p,s}$) rather than more general spaces. It has been reinforced by the now-predominant preoccupation with nonlinear differential equations. Yet use of Fréchet spaces and also of inductive or projective limits of Fréchet spaces is unavoidable and remains alive in other areas of analytic geometry. It suffices to mention Serre's duality in the cohomology of the Dolbeault differential complex.

Granted that Schwartz might have been replaceable as the inventor of distributions, what can still be regarded as his greatest contributions to their theory? This writer can mention at least two that will endure: (1) deciding that the Schwartz space S of rapidly decaying functions at infinity and its dual S' are the "right" framework for Fourier analysis, (2) the Schwartz kernel theorem.

At first the student might not appreciate the full significance of the choice of S, perhaps not realizing that there are many other spaces stable under Fourier transform, spaces of functions that decay much faster at infinity, and not appreciating the deeper fact that, owing to the underlying uncertainty principle, the temperedness of tempered distributions ensures the localizability of their Fourier transform (think of analytic functionals whose Fourier transform can grow exponentially and which are "located" neither here nor there). Certain spaces of Gevrey ultradistributions are also localizable, but although useful in a number of technical questions, they are less "natural".

The Schwartz kernel theorem states a fairly miraculous property of the main distribution spaces, the fact that in certain aspects they are more like finite-dimensional Euclidean space than like infinite-dimensional Banach spaces. In any one of them, C^∞, C^∞_{comp}, $\mathcal{D}'$, $\mathcal{D}'_{\mathrm{comp}}$, $\mathcal{S}$, $\mathcal{S}'$, etc., every closed and bounded set is compact. Moreover, just as the (bounded) linear operators $\mathbb{R}^m \to \mathbb{R}^n$ are elements of $\mathbb{R}^m \otimes \mathbb{R}^n$ (i.e., $m \times n$ matrices), the bounded linear operators, say $K : C^\infty_{\mathrm{comp}}(\mathcal{M}_2) \to \mathcal{D}'(\mathcal{M}_1)$ ($\mathcal{M}_1$ and $\mathcal{M}_2$ are two smooth manifolds), are in one-to-one correspondence with distributions $K(x, y)$ in the product manifold $\mathcal{M}_1 \times \mathcal{M}_2$. With a half-century delay, this gives legitimacy to the general Volterra concept:

$$\forall \varphi \in C^\infty_{\mathrm{comp}}(\mathcal{M}_2), \quad K\varphi(x) = \int_{\mathcal{M}_2} K(x, y)\varphi(y)\,dy,$$

using the physicists' integral notation to mean the duality bracket. Under suitable hypotheses on supports and partial regularity, this gives also a general meaning to Volterra composition: the "kernel" of the composite $K_1 \circ K_2$ is the "integral" $\int K_1(x, y)K_2(y, z)\,dy$. To round off the analogy with the finite-dimensional situation, it must be mentioned that this property is equivalent to the isomorphism of $\mathcal{D}'(\mathcal{M}_1 \times \mathcal{M}_2)$ with the tensor product $\mathcal{D}'(\mathcal{M}_1) \hat{\otimes} \mathcal{D}'(\mathcal{M}_2)$, where the hat indicates completion (in the sense of every "natural" topology on the tensor product): distributions $u(x, y)$ in $\mathcal{M}_1 \times \mathcal{M}_2$ are equal to *infinite* sums $\sum_{n=0}^\infty v_n(x) \otimes w_n(y)$. In all this $\mathcal{D}'$ can be replaced by any one of the other functional spaces above (and many others). The Schwartz kernel theorem was Grothendieck's starting point in building his theory of *nuclear spaces* (the kernel theorem is true because the spaces under consideration are nuclear). Today there is no real need to know the proof of the Schwartz kernel theorem (there is a very simple proof due to L. Ehrenpreis). The theorem provides the foundation on which to start studying special classes of operators, for example, pseudodifferential operators or Fourier integral operators, by studying the corresponding kernel distributions.

Today it is hard to conceive of pseudodifferential operators or of Fourier integral operators without distributions as defined by Schwartz in 1945. More generally, their definition provided the language for vast tracks of mathematics, pure and applied.

It is also hard to reconstruct how difficult it was to arrive at the right definition (many other definitions have been devised subsequently, but none has passed the test of time). It is in the nature of mathematics that most of its theorems and definitions are destined to be simplified and, at least for the very successful, to come to seem obvious and to be taken for granted. Dieudonné used to say that mathematicians would like to be remembered for their most difficult theorems, but most of the time it is their simplest results that survive in the collective consciousness of later generations.

To close this section, the following anecdote might be illustrative. In 1948 Laurent Schwartz visited Sweden to present his distributions to the local mathematicians. He had the opportunity of conversing with Marcel Riesz. Having written on the blackboard the integration-by-parts formula to explain the idea of a weak derivative, he was interrupted by Riesz saying, "I hope you have found something else in your life." Later Schwarz told Riesz of his hopes that the following theorem would eventually be proved: every *linear* partial differential equation *with constant coefficients* has a fundamental solution (a concept made precise and general by distribution theory). "Madness!" exclaimed Riesz. "This is a project for the twenty-first century!" The general theorem was proved by Ehrenpreis and Malgrange in 1952. At the end of the twentieth century, there were proofs of it in ten lines.

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Laurent Schwartz, Radonifying Maps, and Banach Space Geometry

Gilles Pisier

Although he did not himself work in that direction, Laurent Schwartz had a major influence on the rapid development of the geometry of Banach spaces in the early 1970s, the expansion of which continues till today.

This happened mainly through a series of yearly seminars starting with the 1969–1970 "red seminar" on Radonifying maps (see [11]), followed by the eight volumes of the Maurey-Schwartz functional analysis seminars from 1972 to 1981 (with a one-year gap 1976–77 due to Maurey spending that year visiting the Jerusalem Institute for Advanced Studies). At the University of Paris the tradition of publishing the notes of "influential seminars" was well established at that time (notably around members or ex-members of the Bourbaki group), but these notes usually appeared after a long delay, because the university was notoriously understaffed (the mathematics department was reduced then to the Institut H. Poincaré). From 1969 on Schwartz was able to use the considerably richer resources of the École Polytechnique to publish one or two seminar volumes per year. He had created the Centre de Mathématiques there in 1965, and that center published an impressive number of seminar volumes on partial differential equations (in collaboration first with Goulaouic, later with other specialists). During the period 1970–72 the latter seminar included a few "guest lectures" in functional analysis (notably one by Kwapień on Enflo's counterexample to Grothendieck's approximation problem and one by Pietsch on p-absolutely summing maps), but after that an independent seminar was started with Maurey (later on, Beauzamy and the author became virtual co-organizers). The notes for a given talk were distributed usually a few weeks later and were gathered and reproduced without delay at the end of the year. Each of these seminars attracted to the École Polytechnique numerous visitors and played a "federative role": they quickly became a precious meeting point for the different directions inside their subject.

Gilles Pisier is professor of mathematics at Texas A&M University and at Université Paris VI. His email address is `pisier@math.tamu.edu`.

While still at the university, Schwartz himself had published the notes of a classical seminar devoted to making more accessible Grothendieck's thesis on tensor products of locally convex spaces, which he had supervised [17]. However, when Schwartz started his theory of Radonifying maps, I doubt that he foresaw that it would relate back to Grothendieck's "Résumé" [3], as it later did through Saphar's and Maurey's theses. Schwartz's motivation was rather the study of probability theory on infinite-dimensional spaces. It is well known that he had had a keen interest in Brownian motion early on [18] (perhaps helped by his family ties: Paul Lévy was his father-in-law). So the initial motivation came mostly from measure theory. In the mid-1960s he had given a course on Radon measures on topological spaces at the Bombay Tata Institute that was later published [21]. The 1960s also saw the emergence as such of the theory of "infinite-dimensional Gaussian measures" or equivalently of "Gaussian processes" (among which Brownian motion is the classical example). We have in mind the works of Gelfand-Vilenkin, Sazonov-Minlos (for measures on nuclear spaces), Leonard Gross (for "abstract Wiener spaces"), Richard Dudley, Xavier Fernique, Michael Marcus, Larry Shepp (for continuity and integrability of Gaussian processes), and Jean-Pierre Kahane (for random Fourier series [5]), to name a few whose work must have been inspirational to Schwartz at some point.

The following problem (originating in Kolmogorov's work) occupied center stage: Given a Banach space E (say separable for simplicity), consider a random process $\{X(t,\omega) \mid t \in E^*\}$ such that $t \to X(t,\cdot) \in L_0(\Omega, \mathcal{A}, P)$ is linear and continuous. When do the "random paths" $t \to X(t,\omega)$ correspond ω-almost surely to an element of E? In other words, when is there a (strongly measurable) E-valued random variable $\omega \to e(\omega)$ such that $X(t,\omega) = \langle t, e(\omega)\rangle$ for all t in E^*?

For example, when $E = C([0,1])$ (the space of continuous functions on $[0,1]$), and

$$\forall \mu \in C([0,1])^* \quad X(\mu) = \int X_t \, d\mu(t),$$

we are equivalently asking, When does the process $(X_t)_{t\in[0,1]}$ have (a version with) continuous paths?

Another example is provided by Gaussian measures: Let (g_n) be a sequence of independent standard Gaussian random variables (with mean 0 and variance 1). Let (x_n) be a sequence in E such that

$$\forall t \in E^* \quad \sum_n |t(x_n)|^2 < \infty.$$

Then, if we define $X(t,\omega) = \sum_n g_n(\omega)\langle t, x_n\rangle$, we find (by a 1968 result of Itô-Nisio; see, e.g., [4]) that the above question is equivalent to, When does the series

$$\sum_n g_n(\omega)x_n$$

converge strongly in E for almost every ω?

Actually, Schwartz preferred to think of this problem using cylindrical measures instead of linear random processes. One may associate to the linear process X the collection

$$\lambda = \{\lambda_S \mid S \subset E^*, S \text{ finite}\},$$

where λ_S is the probability distribution on $\mathbb{R}^S$ of the restricted process $\{X(t,\cdot) \mid t \in S\}$. Conversely, any family of probabilities $\{\lambda_S\}$ (satisfying an obvious linearized consistency requirement) comes from a process X as above. When the answer to the above problem is positive, the associated "cylindrical probability" on E (meaning the projective system $\lambda = \{\lambda_S\}$) actually comes from a Radon probability on E, the distribution of the above random variable $\omega \to e(\omega)$. One can then essentially say that λ "is" Radon if and only if the answer to the above problem is positive. For instance, if E is a Hilbert space, then, in the second example above, λ "is" Radon if and only if $\sum_n \|x_n\|^2 < \infty$.

The Radonifying maps are linear maps $u : E \to F$ between Banach spaces having the property of transforming a certain class of cylindrical measures on E into Radon measures on F. More precisely, $u : E \to F$ is called p-Radonifying $(0 < p < \infty)$ if for any probability space $(\Omega, \mathcal{A}, P)$ and any continuous linear map $X : E^* \to L_p(\Omega, \mathcal{A}, P)$ the composition $X \circ u^* : F^* \to L_p(\Omega, \mathcal{A}, P)$ comes from a Radon measure on F. Schwartz showed that for the latter measure, $x \to \|x\|_F^p$ is then automatically integrable, so that p-Radonifying maps form a Banach space when $p \geq 1$.

To illustrate this, the continuity of the paths of Brownian motion can be seen as a consequence of the fact that the natural inclusion map $u : \mathrm{Lip}_\alpha \to C([0,1])$ is p-Radonifying when $\alpha > 1/p$ (here $f \in \mathrm{Lip}_\alpha$ means that there is a constant c such that $|f(s) - f(t)| \leq c|s-t|^\alpha$ for all s and t in $[0,1]$).

During the same period, other researchers were studying linear maps with similar properties, notably Vershik and Sudakov in the Soviet Union (see [25]), but whether there was any mutual influence is unclear.

Actually, Schwartz was initially interested in a more general kind of Radonifying map (see [21]), but when informed (apparently through a letter from Kwapień) of the existence of Pietsch's theory of p-absolutely summing maps, he focused on the p-Radonifying ones. At an important meeting that he attended in Poland in 1969 (see [19]), Schwartz met notably S. Kwapień and A. Pietsch and invited them to visit him in Paris. By that time he knew that p-Radonifying maps were almost the same as Pietsch's p-summing maps (which had their roots in Grothendieck's work, at least for $p = 1$ and $p = 2$). He had also determined precisely when a diagonal multiplication $u : \ell_q \to \ell_r$ is p-Radonifying for $0 < p < 1$ and for arbitrary values of $0 < q, r \leq \infty$.

A map $u : E \to F$ is called p-absolutely summing (in short, p-summing) if any sequence (x_n) in E such that $\sup\{\sum_n |\xi(x_n)|^p \mid \xi \in E^*, \|\xi\| \leq 1\} < \infty$ is transformed by u into one such that $\sum_n \|u(x_n)\|^p < \infty$. Consideration of the cylindrical probability associated to (say)

$$\lambda = \sum_{n \geq 1} 2^{-n} \delta_{2^{n/p} x_n}$$

(equivalently of the linear process $X : E^* \to L_p(E, \lambda)$ defined by $X(t, \omega) = \langle t, \omega \rangle$) shows that since

$$\int \|x\|^p u(\lambda)(dx) = \sum_n \|u(x_n)\|^p,$$

p-Radonifying implies p-summing for any $0 < p < \infty$. Schwartz (see [20]) showed that the converse holds for $1 \leq p < \infty$ (assuming F reflexive if $p = 1$). In the case $0 < p < 1$, he needed to assume moreover that E has Grothendieck's metric approximation property, but a counterexample [15] later showed that the result may fail very strongly without this assumption.

Among the first successes to the credit of his theory, the initial proof by Maurey of the "Pietsch conjecture" (independently proved by S. Chevet) used Radonifying ideas. This is a striking result: whenever $0 < p, q < 1$, p-summing and q-summing are the same!

Schwartz's ideas were pursued notably by A. Badrikian and S. Chevet (see [1] and [2]), but once the connection was made with p-summing maps, Maurey's thesis [10] (following H. P. Rosenthal's work on subspaces of L_p [16]) opened up a broad new area of research, and the term "p-Radonifying" was quickly dropped for "p-summing".

Nevertheless, the connection with probability theory (notably Gaussian or q-stable measures for $0 < q < 2$) remained crucial and definitely can be traced back to Schwartz's work. His "duality theorem" for p-Radonifying maps played a key role (see [6]) in later developments. Stated in modern terms, this essentially says that if E^* embeds isomorphically in an (abstract) L_p-space, then every p-summing $u : E \to F$ has a p-summing adjoint $u^* : F^* \to E^*$. This extends the self-duality of Hilbert-Schmidt maps between Hilbert spaces. Moreover, it was later proved (see, e.g., [8]) that the sufficient condition that E^* embeds in L_p is actually necessary. For example, this applies when

$E^* = L_2$ for any p (Gaussian embedding) or when $E^* = L_q$ with $0 < p < q < 2$ (q-stable embedding).

Undoubtedly, this result was part of Maurey's initial inspiration when he introduced "type", "stable type", and "cotype" for Banach spaces: A Banach space E is called of type p if $\sum_n \|x_n\|^p < \infty$ implies the almost sure convergence of $\sum_n g_n x_n$ (where the random variables (g_n)'s are Gaussian as above). It is called of cotype p if the converse implication holds. Shortly afterwards, in 1974, combined work due to Krivine, Maurey, and the author led to geometric characterizations of spaces admitting type (or cotype) "better" than $p \neq 2$ (see [12]).

From that point on, these notions quickly led to major advances in the "geometry" of Banach spaces. This is what Pełczyński called "the great French revolution in Banach space theory" in his plenary address to the 1983 ICM in Warsaw [14]. For instance, the close connection of these notions with the dimensions of the almost-spherical sections in Dvoretzky's famous theorem had a huge impact (see [13] and the very recent survey [12] for more details and references). Type and cotype remain fundamental for anyone interested in Banach spaces, notably in "local theory", i.e., the part of the theory that concentrates on asymptotically large but finite-dimensional, often geometrical, considerations.

Maurey was also stimulated by Hoffmann-Jørgensen's work [4] (brought to his attention by Kwapień), as well as by Kwapień's isomorphic characterization [7] of Hilbert spaces as those that have both type 2 and cotype 2. Hoffmann-Jørgensen (himself greatly influenced by Kahane's book [5]) had independently introduced (apparently slightly earlier) similar notions of type and cotype, but he was mainly motivated by probabilistic considerations. In addition to its impact on "local theory", the convergence of his work with Maurey's is at the root of an explosion of results on "probability on Banach spaces" during the period 1975–81, when type and cotype turned out to be the key to the study of the law of large numbers, the central-limit theorem, or the LogLog law on Banach spaces (see, e.g., [9]).

In retrospect, it seems type and cotype provided the sort of "classification" of Banach spaces that came just at the right time and was exactly suited for the needs of (at least) two distinct groups of researchers: on the one hand, those interested in geometry (or structure) and, on the other, those interested in probability. In addition, it provided a new framework for Banach-space-valued harmonic analysis.

Throughout the 1970s and in the many countries that he visited, Schwartz's support of these new developments was invaluable: Rather than present his own research (about disintegration, Markov processes, or martingales), he often preferred to enthusiastically survey the "geometry of Banach spaces", building on his knowledge of his students' work (see [22], [23], [24]). In 1968 the "new" Banach space theory was already growing strongly in Poland (Pełczyński) and in Israel (Lindenstrauss); later it caught on in the U.S. (H. P. Rosenthal, P. Enflo, W. B. Johnson,…), but the support of a mathematician as famous as Schwartz was nevertheless extremely precious. At that time, although the theory of distributions was, of course, very highly respected, the theory of locally convex spaces had lost its appeal, and the mathematical "establishment" seemed mostly skeptical (to say the least) that going "back" to Banach spaces would prove fruitful. Three decades later that reluctance has surely faded, the field has produced two Fields Medalists, and a lot has happened (see the 2,000 pages of the *Handbook* in reference [12]). As usual, this is the result of a vast collective enterprise, sometimes with no connection to Schwartz, but we find it quite appropriate to acknowledge here the debt that the subject owes him.

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Deux maîtres es probabilités: Laurent Schwartz et Paul-André Meyer

Marc Yor

It is at the same time a sad occasion and a great honor for me to evoke some of the achievements of Laurent Schwartz in probability theory in general and his research on stochastic processes in particular.

Although I shall mainly discuss the works of Laurent Schwartz, I find it impossible not to associate the name and works of Paul-André Meyer to this *évocation*: indeed, not only did the death of Paul-André Meyer occur on the 31st of January 2003, only a few months after the death of Laurent Schwartz on the 4th of July 2002, but their works, as far as the topics of stochastic differential geometry and (in part) stochastic integration are concerned, were developed in close connection, each one having read and commented on the other's work prior to the publication of his own work.

More precisely, Meyer's *Cours sur les Intégrales Stochastiques* [7], which appeared in 1976, was a landmark in the deep understanding of the meaning of the general (real-valued or vector-valued) semimartingale X and of the stochastic integrals $\int H\,dX$ which one can associate to X. Indeed, by the end of the 1970s semimartingales were *characterized* independently by Bichteler and Dellacherie-Mokobodzki, following much pioneering work by Métivier-Pellaumail, as the "good" integrators of bounded predictable processes. For a remarkably concise and informative presentation of stochastic calculus, see Meyer's appendix to Michel Émery's volume [4] *Stochastic Calculus in Manifolds*.

This brings us back to Laurent Schwartz's contributions to stochastic calculus on manifolds: one of the motivations of his work in this domain, very clearly presented in the introduction to [12], was his remark that since real-valued (or vector-valued) semimartingales are stable under composition by C^2 functions, it should be possible to define a semimartingale X taking values in a differentiable manifold V of class C^2 as a process X such that for every C^2 function $\phi : V \to \mathbf{R}$, $\phi(X)$ is a real-valued semimartingale. This is indeed possible, and, in particular, this notion is stable under C^2 mappings from one manifold to another.

Following this successful definition, it was then tempting to try to define a conformal martingale X taking values in a manifold V as a process such that for every holomorphic function $\phi : V \to \mathbf{C}$, $\phi(X)$ is a conformal martingale. (*Note*: The notion of conformal $\mathbf{C}$-valued martingales was introduced by Getoor and Sharpe (1972) as the class of $\mathbf{C}$-valued, continuous local martingales M such that for every holomorphic function $\phi : \mathbf{C} \to \mathbf{C}$, $\phi(M)$ remains a local martingale; in fact, it suffices that M and M^2 are local martingales. This notion led to the proof that the dual of H^1 is BMO for real-valued martingales; this result was obtained by Getoor and Sharpe in the continuous case and very soon extended by Meyer—without using the concept of a conformal martingale—to the general discontinuous case.)

If V is not a Stein manifold, however, it admits too few holomorphic functions for the above attempt of defining a V-valued conformal martingale to be meaningful. This led Schwartz to a vast discussion of localization procedures, definitions, etc. … Finally, the "right" definition of a conformal martingale X on V is a process such that for every function $\phi : V \to \mathbf{C}$ of class C^2 that is holomorphic on an open set V' of V, $\phi(X)$ is a semimartingale

Marc Yor is professor of mathematics in the Laboratoire de Probabilités et Modèles Aléatoires, Université Pierre et Marie Curie (Paris VI).

that is equivalent on $X^{-1}(V')$ to a conformal martingale.

Of course, one should not be surprised that in his research on stochastic processes, as in his famous work in analysis, Schwartz looks for *weak* definitions to "free" himself from the rigidity of *strong* ones. Again, this leads him to define the interesting notion of a formal semimartingale [13]: there, it is the differential symbol (dX_t) (more accurately, this symbol is defined in [17]), or the family of integrals $\int H\,dX$ for suitable H's, that has some meaning and not (X_t) itself.

It is noteworthy that, in doing so, Schwartz obviates the problem of looking for some suitable extension of the class of semimartingales, such as (various types of) Dirichlet processes (as introduced by Föllmer): although nowadays a large family of such processes, mainly obtained as functionals of Brownian motion, has been identified, no corresponding unifying theory of stochastic integration has emerged.

Besides the introduction of formal semimartingales, Schwartz's discussion of semimartingales valued in manifolds, together with the associated theory of stochastic integration and resolution of stochastic differential equations (SDE), may be condensed in what both Meyer [8] and Émery [4] call Schwartz's principle: If one writes Itô's formula as

$$d(f(X_t)) = (d\underline{\mathrm{X}}_t)f$$

with (Schwartz's notation)

$$d\underline{\mathrm{X}}_t = dX_t^i(\omega)D_i + \frac{1}{2}d[X^i, X^j]_t(\omega)D_{ij} \tag{1}$$

for $f : V \to \mathbf{R}$ or $\mathbf{C}$ a C^∞ function and (x^i) global coordinates on V, one is led to understand the $d\underline{\mathrm{X}}_t$ "pseudo-mathematical" object as a tangent vector of second order, which, in fact, does not depend on the choice of coordinates.

Meyer [8, p. 257] prefers to write (1) as d^2X_t and to call it the speed of X (and not its acceleration!). This then leads Schwartz and Meyer to consider stochastic integrals as second-order objects by defining the integrals along the paths of a semimartingale X of C^∞ forms of order 2 on V; precisely, if

$$\theta = a_i\, d^2x^i + a_{ij}\, dx^i dx^j,$$

then

$$\int_{X_0^t} \theta \overset{\text{def}}{=} \int_0^t a_i(X_s)\, dX_s^i + \frac{1}{2}\int_0^t a_{ij}(X_s)\, d\langle X^i, X^j\rangle_s,$$

which in agreement with Schwartz's principle may be presented as

$$\int_0^t \langle d\underline{\mathrm{X}}_s, \theta\rangle.$$

It is noteworthy that, although Schwartz discusses conformal martingales on manifolds [12], he does not really discuss the notion of a martingale, which is in fact done by Bismut (see Meyer [8, p. 258]; W. Kendall pointed out to me the relevant *Comptes Rendus* note of Bismut [1]): here, both a connection Γ on V and a filtration on the probability space are needed to define a Γ martingale on V, whereas the definition of a conformal martingale does not necessitate any connection.

By insisting systematically on the deep geometric (intrinsic) meaning of semimartingales and their stochastic integration and stochastic differential equations, Laurent Schwartz has given a tremendous help to the probabilists' community to learn and "integrate" some (!) differential geometry into their common working tool kit, thus making them able to understand better and generalize beyond the Markovian case Itô's early works on SDEs in a differentiable manifold (Nagoya (1950), Kyoto (1953)) and his stochastic parallel displacement (Stockholm (1962); 1975). I would also like to mention Itô's discussion of stochastic differentials (number 37 in Itô's list of publications; see Itô's selected papers [5] for all the references to Itô's works given here).

Anyone who wishes to study the different aspects of the subject (of stochastic parallel displacement, in particular) in earnest ought to read the very informative and excellent Lecture Notes volume of D. Elworthy [3], who even compares the Bismut-Meyer-Schwartz general theory (p. 179) and the deep results of Itô and Malliavin (see, e.g., [6]). For a treatment with a different flavor and many examples, see Rogers-Williams [11, Chap. V].

While discussing the existence and uniqueness of solutions of stochastic differential equations, Schwartz also made some fine remarks about the speed of convergence of Picard's series: $\sum_n \|X^{(n+1)} - X^{(n)}\|_2 < \infty$, where $X^{(n)}$ denotes the classical nth iterate of a process through Picard's "machine" associated to an SDE with Lipschitz coefficients. See the original paper [18] and the discussion in Dellacherie-Maisonneuve-Meyer ([2, p. 360 et seq.]).

I could witness personally (I have in mind here the paper [16], whose topic is closely linked with Brownian bridge processes, which used to intrigue Schwartz a lot) that Laurent Schwartz was very much of a perfectionist concerning his work in probability (and, no doubt, also in general), asking specialists he thought would help him solve some particular question, showing them early drafts, and so on.

He has been, and so has Meyer, "un modèle pour nous tous," as Alain Connes wrote recently.[1]

[1] *Hommage à Laurent Schwartz,* Gazette des Mathématiciens *(Octobre 2002), no. 94, 7–8.*

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