Correspondance Grothendieck-Serre

Reviewed by Michel Raynaud

Correspondance Grothendieck-Serre
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This volume contains more than eighty letters, mostly written between 1955 and 1966. Today scientists widely use the Internet, and their communication has become disposable as never before. Hence it is necessary to salute the publication of this book. It has been made possible through the initiative of P. Colmez and J-P. Serre. To welcome it, the Société Mathématique de France has started a new series, Documents Mathématiques. Warm thanks are in order.

The period covered by these letters witnessed a prodigious development of algebraic geometry, and the two authors were major contributors. In reading, one is present at the bountiful creation of new ideas and questions. At times the two compatriots move in the same direction, at times in the opposite; sometimes they clash. Overly optimistic conjectures are rejected from one letter to the next; on the contrary, others are refined, strengthened, and…proven. This friendly and unsparing joust in which two grand masters give their all is thrilling.

"My dear Serre …", "Dear Grothendieck…". We are in 1955. Serre is about to be named professor at the Collège de France. Grothendieck is in the United States, having abandoned topological vector spaces and learning topology and geometry, but, he says, not yet engaged in research. The central theme of the discussion is the cohomology of sheaves, either in the topological or complex analytic or algebro-geometric context. What is the best definition of this cohomology? What tools permit its calculation? These questions are rapidly clarified: R. Godement gives a simple method for constructing injective resolutions (his book *Théorie des Faisceaux* will appear in 1958); H. Cartan discovers the spectral sequence associated to a covering. In passing, let us recall that the notions of sheaf and of spectral sequence are due to J. Leray. Serre puts the final touches to "Faisceaux algébriques cohérents" [1], which gives rise to an interesting discussion of duality. Then he continues with "GAGA" [2]: the equivalence between coherent sheaves in algebraic geometry and in analytic geometry in the context of projective varieties. On his part, Grothendieck is already interested in categories and forges his cohomological convictions. He is going to publish "Sur quelques points d'algèbre homologique".

Starting in 1958, Grothendieck, in collaboration with J. Dieudonné, undertakes the redaction of the *Eléments de Géométrie Algébrique* (cited as EGA); the following year he is appointed professor at the Institut des Hautes Études Scientifiques, and in 1960 he launches his *Séminaire de géometrie algébrique* (cited as SGA). Initially he is very optimistic and thinks EGA will be completed in three or four years. It will take noticeably longer and will eventually lapse. But twelve years later—if one puts end-to-end the EGA, the SGA, and his numerous

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exposés at Séminaire Bourbaki—one must acknowledge that he has essentially fulfilled the ambitious program that he embarked upon.

Let us return to 1958. Questions concerning coverings abound. Serre works on pro-algebraic groups and revisits class field theory. Grothendieck reflects on the fundamental group in the theory of schemes. He senses that he needs a "formal GAGA" (which will be included in EGA III). He discovers, in passing, the intimate connections between the properly understood theory of local rings and projective geometry. This will be the unifying theme of SGA 2.

In 1959 a thunderclap: B. Dwork, by a completely unexpected p-adic analytic method, proves the first of the Weil conjectures for finite fields: the rationality of the L-functions. This is a surprise and arouses great curiosity. It will be the subject of Serre's next course at the Collège. Dwork is in a sense ahead of his time. One will wait many years for the development of the p-adic differential calculus in order that his proof be inserted in a general cohomological theory "à la Grothendieck".

Small quarrel apropos valuations and N. Bourbaki. Grothendieck does not like valuations and minimizes his use of them. He reproaches Bourbaki's classicism for having devoted an entire chapter to them. Serre, without being an unconditional enthusiast for valuations, defends Bourbaki's choice.

In 1961 France is fully engaged in the Algerian War. Grothendieck is upset that young beginners, hardly having embarked on research, must devote two years to military service. Serre deplores this also but finds it unjust that one can envision dispensations for intellectuals.

After this political-military episode we return to mathematics. Coherent sheaves are far behind us, and our preoccupations are more arithmetical. Here are two examples, among many, where the sharing of ideas works marvelously. Grothendieck thinks about a Riemann-Roch formula for étale sheaves on algebraic curves. In characteristic $p > 0$, to take into account wild ramification, he must introduce new terms, and he gives a list of the formal properties that they need to satisfy. Serre, familiar with higher ramification and Brauer characters, recognizes there the presence of a particular projective representation of the inertia group. This is the birth of the Swan conductor, which after forty years is still the joy of vanishing-cycle enthusiasts.

Assuredly, 1964 is a good year for the study of the reduction of an algebraic variety defined over a local field. Abelian varieties give a test case, thanks to the initial work of S. Koizumi and G. Shimura, followed by that of A. Néron and A. Ogg. Serre conjectures that an abelian variety, after a finite extension of the local field, reduces to an extension of an abelian variety by a torus. Then D. Mumford proves this, provided the residual characteristic is not 2, thanks to his new approach to theta functions; the semistable reduction of curves intervenes discretely. Finally, Grothendieck establishes that an open subgroup of the inertia group acts unipotently on étale cohomology, then shows that this action can be filtered in at most two steps in the case of curves. From now on number theorists will no longer need to ignore the bad places—indeed, quite the contrary. Let us close this slightly technical interlude.

What about "motives"? you ask. This conjectural theory appears in the correspondence of 1964. Grothendieck succinctly explains it to Serre and uses it as an Ariadne's thread, a philosophy, a "yoga". As he progresses in understanding algebraic cycles, motives increase their presence.

The final letters are notably later and date from 1984–85. The situation has changed. Grothendieck has withdrawn his public presence more than twelve years earlier and only intermittently works on mathematics. He has written *Récoltes et Semailles*, a polemical text. He reproaches his former students for having buried him after his departure, all the while pilfering his most beautiful ideas. But he also reproaches them for not having continued along the path that he outlined: "the big work site abandoned." Serre disagrees and thinks that he complacently darkens the picture. The tone is bitter. Grothendieck, in a fully introspective mode, wants to draw his old friend onto this new terrain. Serre resists. Serre tries to return to mathematics and explains his conjecture on Galois representations and modular forms (which will, ten years later, play a big part in the solution of Fermat's problem), but Grothendieck is not seduced.

In rereading these lines, I am aware that my text is but a summary and is superficial. It gives at most indications but has no pretense to render fully the diversity and the depth of the themes discussed.

This book will evoke good memories for the elder mathematicians. Younger mathematicians will see a superb example of loyal and unbridled cooperation, carried out at the highest level. Historians of science will find much first-hand information…, come the day when they engage in the study of this most fertile period.

References

[1] J.-P. Serre, Faisceaux algébriques cohérents, *Ann. of Math.* **61** (1955), 197–278.

[2] ____ , Géométrie algébrique et géométrie analytique, *Ann. Inst. Fourier* **6** (1956), 1–42.