H. S. M. Coxeter (1907–2003)

Erich W. Ellers, Branko Grünbaum, Peter McMullen, Asia Ivić Weiss

Erich W. Ellers

Harold Scott MacDonald Coxeter died on March 31, 2003, after sixty-seven years as professor at the Department of Mathematics of the University of Toronto. Always known as Donald, he was a man with a vision. His whole life was devoted to the discovery and the description of the symmetries that exist in Euclidean spaces of any dimension. He was particularly fascinated by objects in four dimensions as extensions of those in three dimensions, and also by the projections of objects in dimensions even greater than four. The focus on the fourth dimension started for him at a very early age, when he was still a boy in school, and it stayed with him till his death.

Donald was firmly rooted in the tradition of geometry that goes back in history to Euclid and earlier. He made major and fundamental contributions to this subject which have influenced many other branches of mathematics. University of Alberta mathematician Robert Moody wrote:

> Modern science is often driven by fads and fashion, and mathematics is no exception. Coxeter's style, I would say, is singularly unfashionable. He is guided, I think, almost completely by a profound sense of what is beautiful.

Symmetries occur frequently in nature, manifested by the beautiful regular structure of crystals. Symmetries are also abundant in the arts, for instance in frieze patterns. Coxeter was intrigued by these phenomena. All his life long he emphasized the art in science and the artist in the mathematician. Some of his research work is on frieze patterns; some is relevant for the theory of the structure of crystals.

Certain groups of symmetries with particularly nice properties that have been investigated by Donald are now known as Coxeter groups. These are omnipresent in various modern branches of mathematics. Some examples of Coxeter groups can serve as a basic introduction to group theory for high school students. This has indeed been done successfully in 1997 in a University of Toronto summer school under the title "Coxeter's Geometry".

Two pieces of geometric art at The Fields Institute in Toronto were clearly initiated and encouraged by Donald Coxeter. A symmetrical combination of interlocking triangles adorns the front lawn of the institute, and an intricate mobile, a three-dimensional projection of a four-dimensional polytope, hangs from the ceiling in the atrium.

It is not surprising that Donald was a friend of the Dutch graphic artist M. C. Escher. In 1997 Donald published a paper in which he proved that Escher, despite knowing no mathematics, had achieved mathematical perfection in the etching *Circle Limit III*. "Escher did it by instinct," Coxeter explained; "I did it by trigonometry."

Donald donated a portion of his collection of geometric models to the University of Toronto. These models are kept in a display case in the hall of the fourth floor of Sydney Smith Hall.

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Biography

Donald Coxeter was born on February 9, 1907, in London, England. He got his Ph.D. in 1931 at the University of Cambridge. He was a research fellow at Trinity College, Cambridge, from 1931 to 1936, a Rockefeller Foundation Fellow at Princeton 1932–1933, and a J. E. Procter Fellow 1934–1935, also at Princeton. In 1936 he joined the faculty of the Department of Mathematics at the University of Toronto. He stayed there all his life.

Coxeter was welcomed as a visiting professor at prestigious universities in Britain, the Netherlands, Italy, Australia, and the USA. He was an inspiring and popular lecturer at many conferences and universities all over the globe. He pursued this activity throughout his life. In July 2002 he gave an invited address at the conference on hyperbolic geometry in honour of János Bolyai in Budapest, Hungary. Donald had a number of honorary doctorates from universities in Canada (Alberta, Waterloo, Acadia, Trent, Toronto, Carleton, McMaster) and the University of Gießen in Germany.

He was a Fellow of the Royal Society of Canada; a Fellow of the Royal Society (London); a Foreign Member of the Koninklijke Nederlandse Akademie van Wetenschappen; a Foreign Member of the American Academy of Arts and Sciences; and an Honorary Member of the Mathematische Gesellschaft in Hamburg, the Wiskundig Genootschap, and the London Mathematical Society.

He served the mathematical community by being editor in chief of the *Canadian Journal of Mathematics* for nine years, president (Section III) of the Royal Society of Canada, president of the Canadian Mathematical Congress, vice president of the American Mathematical Society, and president of the International Congress of Mathematicians.

In 1995 he received the CRM/Fields Institute Prize, and in 1997 the Distinguished Service Award of the Canadian Mathematical Society and the Sylvester Medal of the Royal Society of London. In the same year, H. S. M. Coxeter was named a Companion of the Order of Canada. The citation says:

> Through his research, he has made a monumental contribution to the study of geometry by furthering its applications in mathematics, science, art, music, architecture, and crystallography. … [He] has influenced generations of teachers and students for more than half a century.

Mathematical Influence

Coxeter contributed greatly to our mathematical knowledge. He had over two hundred scientific publications to his credit. In 1970 Coxeter wrote an article on "Solids, geometric" for the *Encyclopaedia Britannica*. He also contributed sections to many books and proceedings. Donald is widely quoted; MathSciNet shows five hundred items with "Coxeter" in the title.

Books by H. S. M. Coxeter

Generators and Relations for Discrete Groups (with W. O. J. Moser)
Geometry Revisited (with S. L. Greitzer)
Kaleidoscopes (selected writings)
Introduction to Geometry
Non-Euclidean Geometry
Projective Geometry
The Real Projective Plane
Regular Complex Polytopes
Regular Polytopes
The Fifty-Nine Icosahedra (with P. Du Val, H. T. Flather, and J. F. Petrie)
Twelve Geometric Essays (reprinted as *The Beauty of Geometry*)
Twisted Honeycombs (with Asia Ivić Weiss)
Zero-Symmetric Graphs (with Roberto Frucht and David L. Powers)

His book *Regular Complex Polytopes* contains advanced mathematical theories of geometric objects, yet it does not fit into any bookcase. It displays many beautiful and sometimes intricate figures. It is meant to be a coffee table book. Coxeter writes in the preface:

> I have made an attempt to construct it like a Bruckner symphony, with crescendos and climaxes, little foretastes of pleasure to come, and abundant cross-references.

Donald was an accomplished musician. He liked to point out relations between music and mathematics.

The book *Geometry Revisited* has been translated into French, Hungarian, and German. Some of the material in this book may be accessible already to a bright high school student. We quote from the introduction:

> … let us revisit Euclid. Let us discover for ourselves a few of the newer results. Perhaps we may be able to recapture some of the wonder and awe that our first contact with geometry aroused.

Coxeter's books have been well received, and several have gone through multiple editions. *Introduction to Geometry* is a textbook for a university course in geometry. It is unique in style, perhaps best described by a remark of Bertrand Russell that Coxeter quotes on a page preceding the body of the text:

> Mathematics possesses not only truth, but supreme beauty—a beauty cold and

Ph.D. Students of H. S. M. Coxeter

John Maurice Kingston (1939)
George P. Henderson (1948)
Gerald Berman (1950)
Lloyd Dulmage (1952)
Seymour Schuster (1953)
William O. J. Moser (1957)
F. Arthur Sherk (1957)
Donald W. Crowe (1959)
Bruce L. Chilton (1962)
William G. Brown (1963)
Cyril W. L. Garner (1964)
Norman W. Johnson (1966)
John B. Wilker (1968)
J. Chris Fisher (1971)
Joseph G. Sunday (1973)
Barry Ross Monson (1978)
Asia Ivić Weiss (1981)

> austere, like that of sculpture, without appeal to any part of our weaker nature…sublimely pure, and capable of a stern perfection such as only the greatest art can show.

This book has been translated into German, Japanese, Russian, Polish, Spanish, and Hungarian. *Projective Geometry* has been translated into Croatian. The titles of the German translations of *Introduction to Geometry* and *Geometry Revisited*, namely, *Unvergängliche Geometrie* (Everlasting Geometry) and *Zeitlose Geometrie* (Timeless Geometry), are very attractive and revealing. They beautifully reflect Coxeter's scientific endeavours and his life's philosophy.

Reminiscences

Donald was the most ardent, the most important, and the most revered member of the Geometry Seminar at the University of Toronto. It used to meet every Tuesday. Donald never missed a seminar and never missed an opportunity to deliver a talk on his ongoing research. He had a wealth of knowledge and liked to bring up tantalizing geometrical questions that seemed new to most people but to which he usually knew the answers.

Coxeter was a vegetarian, he was active in saving the environment, and he promoted peace. His wife, Rien, died three years ago. Since then his daughter, Susan Thomas, has looked after him. He also has a son, Edgar, and several grandchildren and great-grandchildren.

The popularity of a person can perhaps be gauged by the number of anecdotes about him. The press is full of anecdotes on Donald. Here is a typical and very charming one: When his graduate student Asia Ivić Weiss, who is now teaching at York University, told him that she would not be able to come to their regular weekly meetings because she was about to give birth, he gave her a 50-page preprint of a paper. "He said that it was something for me to look through if I had nothing else to do in the labour room."

Entering Donald Coxeter's house in Toronto, you notice an old clock with a motto that Coxeter adhered to: Do not delay, time flies.

Coxeter's Varied Contributions

Branko Grünbaum

Although Coxeter's work on regular polytopes and groups of reflections will probably be viewed as his most important contribution, two other aspects should not be forgotten. One is his unceasing activity on behalf of geometry in general. His many books kept mathematicians aware of the various branches of geometry during much of the twentieth century, at a time when these kinds of geometry were in general decline and in danger of disappearing. In particular, his *Regular Polytopes* in its several editions is possibly one of the most quoted geometry texts of the century. Moreover, the countless reviews he wrote for *Mathematical Reviews* and other publications helped illuminate and place authoritatively in historical context works of many authors.

On the other hand, Coxeter contributed many ideas and methods in studying specific geometric questions. He seems to have been the first to note (in [1]) the equivalence of zonotopes and arrangements of hyperplanes, which nowadays is taken as completely natural. The survey [2] led to increased interest in sphere packings, and [3] introduced new ideas in the treatment of color symmetries. However, one of his most interesting papers is [6], written jointly with M. S. Longuet-Higgins and J. C. P. Miller, and based on results obtained in the early 1930s by Coxeter and Miller, and independently by Longuet-Higgins and his brother in the 1940s; an excerpt was published in [4].[1]

The paper [6] contains a detailed exposition and construction for all uniform polyhedra, that is, polyhedra in which all vertices form one transitivity class under symmetries of the polyhedron and all faces are regular polygons. The convex ones among them are said to have been known to Archimedes; they were (re)discovered by Kepler four centuries ago. There was a trickle of nonconvex

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[1] *B. G. is indebted to M. S. Longuet-Higgins for many details concerning the genesis of [6].*

uniform polyhedra found by various authors starting in the second half of the nineteenth century. However, there was no overall guiding principle in these discoveries. After surveying the then-existing literature and combining the results, the authors of [6] presented a unified method of construction that produced all the previously known ones together with twelve additional polyhedra. The authors of [6] did not claim completeness for their enumeration but expressed hope that it is, indeed, complete. This was shown to be the case some years later, in [10] and [11].

One aspect of the material presented in [6] broke with the entrenched tradition in the study of nonconvex polyhedra and lends importance to the paper that goes far beyond the enumeration itself. Unfortunately, the referees in both *Mathematical Reviews* and *Zentralblatt* did not recognize this. Earlier writers on nonconvex polyhedra (such as A. Badoureau, E. Hess, and M. Brückner) either gave no definition of the class of polyhedra they were considering or gave vague explanations which they themselves ignored. Hence it is obvious that one could not even contemplate any proofs of completeness for any of their enumerations. In contrast, Coxeter et al. gave a precise and reasonable definition, which served well in their work and in later studies of the topic.

The definition in [6] is very simple. "A *polyhedron* is a finite set of polygons such that every side of each belongs to just one other, with the restriction that no subset has the same property.…The faces are not restricted to be convex, and may surround their centres more than once.… Similarly, the faces at a vertex…may surround the vertex more than once. A polyhedron is…*uniform* if its faces are regular while its vertices are all alike." This definition is perfectly suitable for the study of uniform polyhedra; it made possible the proof of completeness of the enumeration. Unfortunately, it is not useful for investigations of more general polyhedra. In later years Coxeter was aware of the problems, which are of two kinds.

On the one hand, if the conditions are weakened, compounds of polyhedra are included as well as various other possibilities which are generally not acceptable (such as several circuits of faces around some vertices). On the other hand, the definition used in [6] admits coplanar faces, and in fact several of the uniform polyhedra have coplanar sets of distinct faces. But following the tradition established by Poinsot and Cauchy in the early nineteenth century (and in contrast to the earlier approach by Meister, which was forgotten for more than two centuries), the polar of this possibility is not admitted. In other words, having two distinct vertices represented by the same point is not permitted under the definitions in [6]. Clearly, this situation makes it impossible to consider the

Donald Coxeter and great grandson Jack Thomas Lions.

polars of the uniform polyhedra. In the attempts to do so, various authors [12], [8] reverted to the earlier custom of disregarding their own definitions. When appropriate and more usable general definitions were developed (in [9] and [7]), Coxeter realized their applicability and encouraged their publication. The possibility of such generalizations is also mentioned in [5]; this exemplifies the adaptability and willingness to consider novel approaches that characterized Coxeter's attitude throughout his life.

In assessing Coxeter's influence, one should not forget his readiness to answer questions and provide advice and references. Many of us are forever indebted to him for helping us in our own research.

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Donald Coxeter and Regular Polytopes

Peter McMullen

The classical description of regular polygons and polyhedra is well known, but the modern theory of regular polytopes begins with Ludwig Schläfli in the middle of the nineteenth century. Schläfli discovered all but six of the regular polytopes in four or more dimensions (he would not recognize the others because—mistakenly—he excluded regular polyhedra of genus other than zero). The regular polytopes were rediscovered by Stringer and others from the 1880s onward; van Oss actually completed their classification.

However, the whole subject of regular polytopes could well have become a mathematical backwater were it not for the work of Donald Coxeter and a circle of like-minded geometers from the 1920s onward. It was Coxeter who introduced the vital connexion with group theory. A *Coxeter group* $\mathcal{G}$ is generated by involutions $R_0, R_1, \ldots, R_{n-1}$ (for some n), satisfying relations solely of the form $(R_j R_k)^{p_{jk}} = E$, the identity. The symmetry group of a regular convex polytope is a *string* Coxeter group, in that $p_{jk} = 2$ whenever $j \le k-2$; the R_j themselves are reflexions in hyperplanes. The polytope is then denoted by the *Schläfli symbol* $\{p_1, \ldots, p_{n-1}\}$, and its group by $[p_1, \ldots, p_{n-1}]$, where $p_j := p_{(j-1)j}$ for $j = 1, \ldots, n-1$. Each such polytope has a *dual*, whose Schläfli symbol is $\{p_{n-1}, \ldots, p_1\}$, and thus obtained by reversal.

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In $\mathbb{E}^n$ we always have three regular polytopes: the *simplex* $\{3^{n-1}\}$, the *cross-polytope* $\{3^{n-2}, 4\}$, and the *cube* $\{4, 3^{n-2}\}$, as well as the *cubic tiling* $\{4, 3^{n-2}, 4\}$, where the notation p^k means a string $p, \ldots, p$ of length k. When $n \ge 5$, these are the only examples. For $n \le 4$, we have other cases as well. Apart from infinitely many regular polygons (the star-polygons included), in $\mathbb{E}^3$ there are the icosahedron $\{3, 5\}$ and dodecahedron $\{5, 3\}$ and the four related star-polyhedra, while in $\mathbb{E}^4$ there are the 24-cell $\{3, 4, 3\}$, the 600-cell $\{3, 3, 5\}$, and the 120-cell $\{5, 3, 3\}$, together with ten star-polytopes related to the latter two.

The polytopes just listed we may refer to as *classical*; they are the main subjects of Coxeter's seminal book *Regular Polytopes*, which went into four editions. This masterly work gives a lucid exposition of the whole theory, not just the polytopes themselves and (where appropriate) how to construct them, but also the related geometry and group theory. I cannot be the only mathematician whose career was deeply influenced by this book.

Coxeter himself provided a main impetus for the development of the theory of regular polytopes in a more abstract direction. As he himself told us, his friend John Petrie discovered two new regular apeirohedra (infinite polyhedra) in $\mathbb{E}^3$: one, $\{4, 6 \mid 4\}$, with six squares fitting round each vertex in a zigzag fashion, and the other, $\{6, 4 \mid 4\}$, with four hexagons around each vertex; the last entry "4" indicates a *hole*, formed by a circuit of edges leaving each vertex by the *second* edge from that by which it entered. Coxeter immediately found a third example in $\mathbb{E}^3$, namely, $\{6, 6 \mid 3\}$, and then other polyhedra whose structure is specified by a Schläfli symbol augmented by a hole. The group of $\{p, q \mid h\}$ is the quotient of $[p, q]$ under the imposition of the extra relation $(R_0 R_1 R_2 R_1)^h = E$; thus the dual of $\{p, q \mid h\}$ is $\{q, p \mid h\}$. For instance, the regular star polyhedra $\{\frac{5}{2}, 5\}$ and $\{5, \frac{5}{2}\}$ are, abstractly, $\{5, 5 \mid 3\}$.

In a similar way, some regular polyhedra are specified by the lengths of their *Petrie polygons*, which are such that two successive edges, but not three, belong to faces of the polyhedra. Such a polyhedron, with p-gonal faces, with q through each vertex, and with Petrie polygons of length r, is denoted $\{p, q\}_r$; the corresponding group is $[p, q]$, with the imposition of the relation $(R_0 R_1 R_2)^r = E$. In all but a handful of cases, the Petrie polygons of $\{p, q\}_r$ themselves form the faces of another regular polyhedron, which is $\{r, q\}_p$. The dual of $\{p, q\}_r$ is $\{q, p\}_r$, and so, with duality, one obtains a family with (in general) six members. As examples, if we identify opposite faces of the icosahedron and dodecahedron, then we obtain the *hemi-icosahedron* $\{3, 5\}_5$ and *hemi-dodecahedron* $\{5, 3\}_5$.

The symmetry of this latter family suggests that the corresponding abstract groups might also be of considerable importance; this is indeed so, but we shall not go into this aspect here (see the section by Asia Weiss). But we should mention the central role played by these groups, and those of the preceding family defined by holes, in the delightful book by Coxeter and Willy Moser, *Generators and Relations for Discrete Groups*.

Coxeter, with others, also contributed to the generalization to abstract regular polytopes of higher rank (the number n of involutions which generate the automorphism group). With Geoffrey Shephard, he found a pretty 3-complex in the 4-sphere composed of 20 solid tori; this actually corresponds to a regular 4-polytope. Further, he discovered abstract regular 4-polytopes whose 11 facets are hemi-icosahedra fitting around its 11 vertices in the manner of hemi-dodecahedra (this was independently found by Branko Grünbaum), and another formed by 57 hemi-dodecahedra fitting around its 57 vertices in the manner of hemi-icosahedra.

In another direction, Shephard defined and enumerated the *regular complex polytopes*. These are close analogues of the real regular polytopes; the generators R_j of their symmetry groups are now unitary transformations of finite period having a (complex) hyperplane of fixed points. This provided Coxeter with the opportunity to produce another famous book, *Regular Complex Polytopes*. Once again, Coxeter displayed his mastery of the subject, resulting in a beautiful exposition; his love of music was also cleverly interwoven into the presentation. (I hope that a personal note is not inappropriate here. The curious format of the book arose from Coxeter's wish to incorporate many of my own drawings of projections of regular polytopes; Coxeter also drew on unpublished results from my qualifying M.Sc. thesis.)

In such a brief survey, I have had, of necessity, to skate over Coxeter's many other contributions to regular polytope theory (I have tried to pluck out a few pearls). What needs to be emphasized is that Coxeter not only consolidated the classical theory but also pointed out several directions in which the abstract theory subsequently developed. His torch is carried on by an enthusiastic group of followers, and his efforts have ensured that the topic of regular polytopes still has plenty of life in it.

Coxeter Groups

Asia Ivić Weiss

A Coxeter group is defined as an abstract group generated by elements ρ_j ($j = 0, \dots, n-1$) subject to relations

$$(\rho_j \rho_k)^{p_{jk}} = E \text{ (the identity)},$$

where $p_{jj} = 1$ and $p_{jk} = p_{kj} > 1$ (possibly ∞) for all $j, k \in \{0, \dots, n-1\}$. These groups play important roles in several branches of mathematics, such as group theory, the theory of polytopes, crystallography, graph theory, Lie groups, and the theory of buildings. J. Tits, who initiated the systematic study of such abstract groups, coined the name based on the pioneering work of Donald Coxeter.

An interest in understanding symmetries of regular and uniform polytopes led Coxeter very early in his life to investigate properties of reflection groups. His approach, combining geometry and algebra, yielded the first comprehensive treatment of reflection groups.

Spherical and Euclidean reflection groups were first studied systematically by Coxeter in [4] and completely classified by him in 1933 [5]. The corresponding classification for the hyperbolic groups is as yet unsolved. For the general theory and the current state of the classification, we refer to papers by Vinberg, for example [11]. F. Lannér completed the classification of hyperbolic reflection groups having a simplex as fundamental domain in his dissertation [9]. In addition, in a natural way the notion of a reflection can be extended to that of a unitary reflection, that is, of a transformation of finite order of a complex Euclidean space with all but one of its eigenvalues equal to 1. The finite unitary reflection groups were classified by Geoffrey Shephard and J. A. Todd in 1954 [10].

Circa 1930, inspired by the work of Todd, Coxeter began to investigate the properties of a group generated by reflections in the facets $R_0, \dots, R_{n-1}$ of a polytope P whose dihedral angle between the jth and kth facets is π/p_{jk}, where the p_{jk} are integers ≥ 2, or 0 when the facets are parallel. First accounts of this can be seen in [2] and also in [1]. (For his work in the two-part paper [1], [3] he received the prestigious Smith Prize, given to the undergraduate student with the best essay on a mathematical topic.)

The polytope P, whose closure forms the fundamental region for the group, can conveniently be denoted by a Coxeter graph: the facets of P are represented by dots and connected by branches, which are labeled by integers p_{jk} so that the mutual

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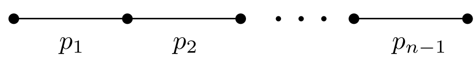
inclination of facets can be read from the graph. A Coxeter graph thus encodes information about the group in a useful manner. More importantly, this representation aids in the classification of reflection groups (see, for example, [7] or [8]). It was during his year at Princeton in 1932–1933 that Coxeter first thought of representing reflection groups by a graph of dots and branches. Independently, but considerably later, the same graphical notation was rediscovered by Dynkin, circa 1940, when he was nineteen years old, in his discussions of simple Lie algebras. A Dynkin diagram is essentially a Coxeter graph with the restriction that $p_{jk} = 3, 4$, or 6 and multiple branches are used in place of labels.

When the fundamental region of a reflection group is an orthoscheme, that is, a simplex whose facets may be ordered so that any two that are not consecutive are orthogonal, the Coxeter graph for the group is a string diagram:



When each $p_j > 2$, the group is the symmetry group of a regular tessellation of spherical, Euclidean, or hyperbolic $(n-1)$-space or an isomorphic n-polytope (see the section by Peter McMullen). In the 1954 paper [6] presented at the International Congress of Mathematicians in Amsterdam, Coxeter gave a complete classification of the hyperbolic tessellations.

Donald Coxeter had a remarkable record of accomplishments. His mathematical contributions spanned over eighty years. At the age of sixteen he won a prize for an essay on the analogues of regular solids in higher dimensions; his last paper was finished the day before his death. I had the privilege of submitting this paper on his behalf to the proceedings of the conference celebrating the 200th anniversary of the birth of the famous Hungarian geometer János Bolyai.

After sending a short notice of Coxeter's death, I received numerous responses from hundreds of mathematicians, some of whom I did not know. Many offered warm reminiscences, but many talked about the inspiration and influence that Donald had on their mathematics. I consider it a privilege to have been his student. I owe him a lot for his guidance and support, but mostly I am grateful for the contagious love of mathematics that he imprinted upon me.

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