WHAT IS... a Curvelet?

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Energized by the success of wavelets, the last two decades saw the rapid development of a new field, computational harmonic analysis, which aims to develop new systems for effectively representing phenomena of scientific interest. The curvelet transform is a recent addition to the family of mathematical tools this community enthusiastically builds up. In short, this is a new multiscale transform with strong directional character in which elements are highly anisotropic at fine scales, with effective support shaped according to the parabolic scaling principle length$^2 \sim$ width.

To fix ideas (although this is a distortion of reality) it is useful to think about curvelets as obtained by applying parabolic dilations, rotations, and translations to a specifically shaped function ψ; they are indexed by a scale parameter a $(0 < a < 1)$, a location b, and an orientation θ and are nearly of the form

$$\psi_{a,b,\theta}(x) = a^{-3/4}\psi(D_a R_\theta(x-b)),$$
$$D_a = \begin{pmatrix} 1/a & 0 \\ 0 & 1/\sqrt{a} \end{pmatrix}.$$

Here D_a is a parabolic scaling matrix, R_θ is a rotation by θ radians, and for $(x_1, x_2) \in \mathbb{R}^2$, $\psi(x_1, x_2)$ is some sort of admissible profile (analogs exist in higher dimensions). The geometry of a curvelet is now apparent: if the function ψ is supported near the unit square, we see that the envelope of $\psi_{a,b,\theta}$ is supported near an a by $\sqrt{a}$ rectangle with minor axis pointing in the direction θ. An important property is that curvelets obey the principle of harmonic analysis stating that it is possible to analyze and reconstruct an arbitrary function $f(x_1, x_2)$ as a superposition of such templates. One can, indeed, easily expand an arbitrary function $f(x_1, x_2)$ as a series of curvelets, much like an expansion in an orthonormal basis. (The question of whether there exist orthonormal bases of curvelets is open.) Continuing at an informal level of exposition, there is a discretization of scale/location/angle which roughly goes like $a_j = 2^{-j}$, $j = 0, 1, 2, \ldots$, $\theta_{j,\ell} = 2\pi\ell \cdot 2^{-\lfloor j/2 \rfloor}$, $\ell = 0, 1, \ldots, 2^{\lfloor j/2 \rfloor} - 1$, and $b_k^{(j,\ell)} = R_{\theta_{j,\ell}}(k_1 2^{-j}, k_2 2^{-j/2})$, $k_1, k_2 \in \mathbb{Z}$, so that with $\psi_{j,\ell,k} = \psi_{a_j, b_k^{(j,\ell)}, \theta_{j,\ell}}$ the collection $(\psi_{j,\ell,k})$ obeys

$$f = \sum_{j,\ell,k} \langle f, \psi_{j,\ell,k} \rangle \psi_{j,\ell,k},$$
$$\|f\|_{L_2}^2 = \sum_{j,\ell,k} |\langle f, \psi_{j,\ell,k} \rangle|^2.$$

All right. So curvelets comprise an interesting new multiscale architecture which gives very concrete representations. There are many others. Why should we care?

Curvelets for What?

It is well known that discontinuities destroy the sparsity of a Fourier series. This is the Gibbs phenomenon; we need many, many terms to reconstruct a discontinuity to within good accuracy. Wavelets, because they are localized and multiscale, do much better in one dimension, but because of their poor orientation selectivity, they do not represent higher-dimensional singularities effectively. What makes curvelets interesting and actually motivated their development is that they provide a mathematical architecture that is ideally adapted for representing objects which display *curve-punctuated smoothness*—smoothness except for discontinuity along a general curve with bounded curvature—such as images with edges, for example. The curvelet transform is organized in such a way that most of the energy of the object is

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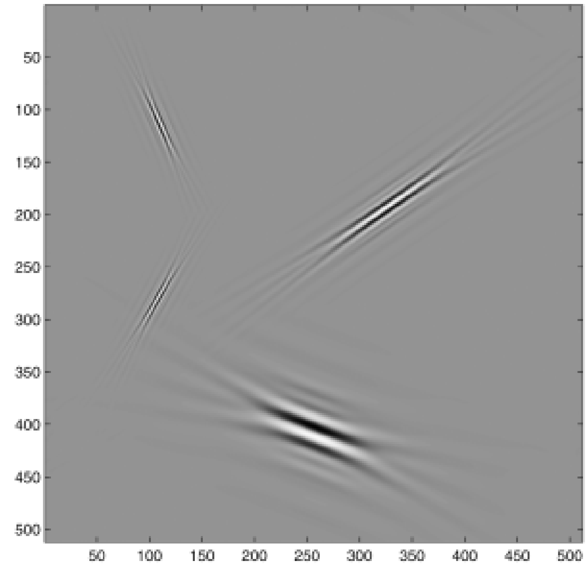
localized in just a few coefficients. This can be quantified. Simply put, there is no basis in which coefficients of an object with an arbitrary singularity curve would decay faster than in a curvelet frame. This rate of decay is much faster than that of any other known system, including wavelets. Improved coefficient decay gives optimally sparse representations that are interesting in image-processing applications, where sparsity allows for better image reconstructions or coding algorithms.

Beyond Scale-Space?

A beautiful thing about mathematical transforms is that they may be applied to a wide variety of problems as long as they have a useful *architecture*. The Fourier transform, for example, is much more than a convenient tool for studying the heat equation (which motivated its development) and, by extension, constant-coefficient partial differential equations. The Fourier transform indeed suggests a fundamentally new way of organizing information as a superposition of *frequency* contributions, a concept which is now part of our standard repertoire. In a different direction, we mentioned before that wavelets have flourished because of their ability to describe transient features more accurately than classical expansions. Underlying this phenomenon is a significant mathematical architecture that proposes to decompose an object into a sum of contributions at different scales and locations. This organization principle, sometimes referred to as *scale-space*, has proved to be very fruitful—at least as measured by the profound influence it bears on contemporary science.

Curvelets also exhibit an interesting architecture that sets them apart from classical multiscale representations. Curvelets partition the frequency plane into dyadic coronae and (unlike wavelets) subpartition those into angular wedges which again display the parabolic aspect ratio. Hence, the curvelet transform refines the scale-space viewpoint by adding an extra element, orientation, and operates by measuring information about an object at specified scales and locations but only along specified orientations. The specialist will recognize the connection with ideas from microlocal analysis. The joint localization in both space and frequency allows us to think about curvelets as living inside "Heisenberg boxes" in phase-space, while the scale/location/orientation discretization suggests an associated tiling (or sampling) of phase-space with those boxes. Because of this organization, curvelets can do things that other systems cannot do. For example, they accurately model the geometry of wave propagation and, more generally, the action of large classes of differential equations: on the one hand they have enough frequency localization so that they approximately behave like waves, but on the other hand they have enough spatial localization so that the flow will essentially preserve their shape.



Some curvelets at different scales.

Research in computational harmonic analysis involves the development of (1) innovative and fundamental mathematical tools, (2) fast computational algorithms, and (3) their deployment in various scientific applications. This article essentially focused on the mathematical aspects of the curvelet transform. Equally important is the significance of these ideas for practical applications.

Multiscale Geometric Analysis?

Curvelets are new multiscale ideas for data representation, analysis, and synthesis which, from a broader viewpoint, suggest a new form of multiscale analysis combining ideas of geometry and multiscale analysis. Of course, curvelets are by no means the only instances of this vision which perceives those promising links between geometry and multiscale thinking. There is an emerging community of mathematicians and scientists committed to the development of this field. In January 2003, for example, the Institute for Pure and Applied Mathematics at UCLA, newly funded by the National Science Foundation, held the first international workshop on this topic. The title of this conference: Multiscale Geometric Analysis.

References

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