Book Review

The Mathematics of Juggling

Reviewed by Allen Knutson

The Mathematics of Juggling
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Around 1985 three groups of jugglers (in Santa Cruz, California; in Pasadena, California; and in Cambridge, England) independently created the same notational system for juggling patterns. These numerical descriptions have been well publicized in the juggling world under the name "siteswaps" (and, even so, continue to be rediscovered by newcomers).

As we will see shortly upon presentation of the basic definitions, a rather simple list of restrictions leads easily to a unique notational system. More surprising is the near-simultaneity of the discovery! The best explanation of this seems to be the huge increase in the number of jugglers worldwide in the 1980s, in no small part due to the publication of *Juggling for the Complete Klutz* [CR] in 1978.

While siteswaps have played a minor role in a number of books on ball juggling (written, naturally, for jugglers), *The Mathematics of Juggling* is the first to put them in the forefront. Before getting to the book per se, I will explain what siteswaps are.

Siteswaps

What are the assumptions on a notational system that led all these people to the same solution? Some of them are restrictions on how the juggler behaves (rather strong restrictions, which we will weaken later):

- The juggler throws only one ball at a time, never holding more than one in each hand.
- The throws alternate hands, right left right left.
- (to avoid considering boundary conditions) The juggler has been juggling since the infinite past and will continue into the infinite future.
- The throws come one per second, with right throws at even times and left throws at odd times (this can be ensured by reparametrizing the time coordinate).

More notable, though, is the assumption about what one actually wants to record:

- The only information kept about the ball thrown at time $n \in \mathbb{Z}$ is when it is next thrown, at time $f(n)$.

In particular, this assumption loses any extra detail about the throw like "under the leg", "with a triple spin" (if the objects juggled are pins rather than balls), "while telling joke #1729", etc. But it suggests a very nice mathematical theory of certain functions $f : \mathbb{Z} \to \mathbb{Z}$:

1. The function f should be one-to-one, since we do not want two balls landing at the same time.
2. The function f should be onto, so that we always have a ball available to be thrown.
3. The function $g(n) := f(n) - n$ might be required to be periodic with period p, since we are most interested in patterns that repeat. This g is really the most natural way to encode f, as its values do not change if we shift the origin in our time coordinate.
4. Since a ball thrown at time n is *next* thrown at time $f(n)$, we should have $f(n) > n$. So each $g(n) > 0$. (In fact one usually wants to weaken this to $g(n) \geq 0$, but we will not get into that.)

This last assumption is the most annoying one for mathematicians, as it forbids the possibility that our family $\{f : g \text{ is periodic with period } p\}$ might form a group—but of course it is firmly insisted upon by jugglers!

For a taste of the theorems to come, the reader is invited to prove the following: if the function g is indeed periodic, then its average is the number of orbits of the permutation f. This is useful to the juggler, who interprets this average as the number of balls in the pattern.

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A "siteswap" then refers to one period of such a g. To g we can associate a permutation living in the symmetric group $Sym(\mathbb{Z}/p\mathbb{Z})$ where p counts the number of "sites" that are then "swapped" by the balls' landing in a different order than they were thrown.

Our main examples are the two best known juggling patterns: the three-ball *cascade* in which all balls are thrown to the same height and all cross, and the three-ball *shower*, or "circle juggling", in which the right hand (for most people) throws high and across and the left hand only shuttles the balls underneath, not really making a throw. The cascade is encoded by $f(n) = n + 3$, since while one ball is in the air the other two balls each get thrown once. The associated siteswap is just "3", or 33, or perhaps 333, really meaning $\ldots, 3, 3, \ldots$. (In practice one hardly ever deals with values of g above 9—these throws are very high and so extremely difficult for a juggler to control—so it is safe to dispense with the commas in between the values of g. One can also use letters to encode throws up to 35, making words like TEAKETTLE and even THEOREM into legal, if humanly impossible, juggling patterns. This amusing idea seems to be first due to Michael Kleber in the 1980s, but of course has been independently discovered since.)

The shower is more annoying to encode as a function f—it is $f(n) = n + 5$ for n even, $f(n) = n + 1$ for n odd—and sounds better as a siteswap 51 (or 15, or 5151, etc.). While this fits with the averaging rule above, the 5 may seem unintuitively high; to see that it is right, note that after the right hand makes such a high throw, the other two balls must each make it through both hands, accounting for four throws in between. So the ball thrown as a 5 is then fifth in line to be thrown again, as it should be.

How much more is there to a siteswap than the associated permutation? If we forget the $g > 0$ restriction and just look at the group $\{f : \mathbb{Z} \xrightarrow{\sim} \mathbb{Z} \mid f(n + p) = f(n) + p, \ \forall n\}$, it turns out to be the affine Weyl group $S_p \ltimes \mathbb{Z}^p$ of $GL(p)$, denoted $\hat{A}_{p-1}$. Therefore, given a formula to count juggling patterns, one can exploit it to compute the Poincaré series of $\hat{A}_{p-1}$ [ER].

As we are now getting into serious mathematics, let us turn to the book under review.

The Contents of *The Mathematics of Juggling*

Having given a great many lectures on the mathematics of juggling, to mathematicians and to jugglers, I was curious at whom this book would be aimed. I think it is fair to say that it is aimed squarely at the mathematician, with juggling patterns (and change ringing for carillons!) to serve primarily as a source of combinatorial problems. Some evidence of this bias will be given below.

Of course, the author can and must make the choice of which audience to address. Because this is the first book dedicated to this topic, it is especially incumbent upon him to get the history right, since any errors are likely to be repeated in subsequent books. I am afraid that in this measure the book is lacking. A few examples: one of the founding fathers of siteswaps, Paul Klimek of Santa Cruz, has his name spelled "Klimak" throughout. The truly fundamental "juggling state graphs", which let one regard a juggler as a sort of finite-state automaton, are nowhere attributed to their creator, Jack Boyce.

Also, much of the terminology has been changed, rather willfully, the most galling change being "siteswap" to "juggling sequence". It is hard enough communicating with people in other disciplines (e.g. physics) who have developed a theory in parallel with mathematicians and thus have different names for the same object. In this happy instance, notational confusion *could* have been avoided, as the jugglers have been using a largely consistent terminology for nearly twenty years.

Much more mysterious than the historical and notational lapses, though, is the lack of actual siteswaps! Occasionally one will find an exhaustive list of the siteswaps fitting a given description, such as the table on page 56 of "maximal prime juggling sequences" of a given length and number of balls. The siteswaps showing up in these extremal-combinatorics tables are essentially unjugglable. (For example, most have a very high throw followed by a very low one, making it impossible for the juggler to see both balls.) As near as I could determine, the book includes only about a dozen siteswaps of any interest to jugglers, and even for those one must wait until Chapter 5.

Let's take a closer look at the contents.

Chapter 2, "Simple Juggling", introduces the definitions touched on above and juggling state graphs. Chapters 3 and 4 loosen the unnecessarily strict conditions we put on juggling patterns: Chapter 3, "Multiplex Juggling", studies patterns in which f is neither one-to-one nor singly valued, but instead for each n the number of balls coming in equals the number of balls going out. Chapter 4, "Multihand Juggling", allows for a set H of hands to throw at the same time, basically replacing $f : \mathbb{Z} \to \mathbb{Z}$ with $f : H \times \mathbb{Z} \to H \times \mathbb{Z}$. This allows for a new concept of "duality" (which I learned about from this book), in which the roles of the hands and the balls are exchanged: according to the balls' point of view, they are sitting still, and the hands are being "thrown" back and forth!

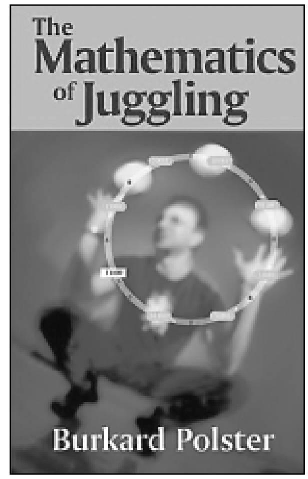
There are many mathematical results contained in these chapters, some with rather involved proofs; I would say that together these are the greatest strengths of the book. For example, in the first chapter a full five pages is spent on a rigorous proof of the following result: given a sequence of natural numbers whose average is an integer, there is some permutation of it that is a siteswap. While the proof is elementary, I suspect that extremely few jugglers without mathematical training will have the fortitude to make their way through five pages of it. I would also claim that this particular theorem is of no intrinsic interest to jugglers (being one myself); this is not a value judgment, but simply indicative of the emphases of the book. Incidentally, some of the theorems are due to the late Claude Shannon, who was also the first to build a juggling robot.

Chapter 5, "Practical Juggling", is perhaps rather jokingly named, given that it spends as much time discussing juggling in zero gravity as it does three-ball siteswaps. Some of the physics is also discussed. For example, if each of a club juggler's hands are full for exactly the same fraction of the time as they are empty and the club juggler uses no wrist flick in order to spin the clubs, then (as first explained by the Caltech founders of siteswap, Bengt Magnusson and Bruce Tiemann, whose analysis is reproduced here) one should expect the clubs in flight to be parallel to one another. For competent jugglers, this lining up is readily observed in photographs, prompting the question (not answered here or, to my knowledge, anywhere else): What goodness function is the juggler maximizing by having each hand full exactly half the time?

In my experience, mathematicians are generally very curious to see how their esoteric theories are actually of direct interest to nonmathematicians (witness, for example, the huge crowd that shows up to any "String theory for mathematicians" lecture). So it is extremely disappointing that the "What's all this good for?" section is only two pages with just one, throwaway, example. (Readers seeking more meat might consult [K].)

Many siteswaps are given odd attributions, though admittedly to determine the provenance of juggling patterns is notoriously difficult. One that struck me was "Allen Knutson's baby-juggling pattern 52512", which presumably was attributed to me because I pointed out on the USENET newsgroup `rec.juggling` [RJ] that this is the pattern always used to juggle babies, as one of the three objects is never thrown high. As one can find movies of jugglers from the 1930s using this pattern to "juggle" a monkey and two balls, I hereby renounce any claim on this siteswap.

At this point comes one of the longest chapters, Chapter 6, "Jingling, or Ringing the Changes", which delves into the combinatorics and group theory associated to change ringing (which in practice apparently involves traveling to a British cathedral; the author estimates that of the approximately 6,000 churches with rings of bells, only about 100 are outside Britain). In this subject the mathematical constraints partly arise from the physical setup of the bells, but mostly from the *Decisions of the Central Council of Church Bell Ringers* [CC]. They are very stringent, leading to a highly constrained mathematical theory, of which much is presented in this chapter. Half a page is spent on a rather tenuous connection to siteswaps (basically, both are related to permutations).



I found it rather maddening that in this chapter there are far more ringing sequences with names attached than there are named juggling patterns in the entire rest of the book. And then ten pages are spent on 3-D stereograms of the Hamiltonian circuits associated to ringing sequences. What is this book's title anyway?

Supplementary Material

The author states his aims in the introduction as Serious Mathematics, Serious Juggling, Mathematics Education, Turn Jugglers into Mathematicians, Turn Mathematicians into Jugglers, and Bell Ringing. I would say he certainly succeeds in the first and the last and am distinctly more reserved about the other aims. For both the jugglers wanting to leverage their knowledge into an appreciation of the mathematics and the mathematicians (young and old) looking to juggling for inspiration, the book could and should have gone much further into the exploration of interesting examples.

There is no shortage of such examples on the World Wide Web; I direct the reader to the Juggling Information Service (the "JIS") at `http://www.juggling.org` and in particular to `http://www.juggling.org/help/siteswap`. Many examples are built directly into juggling animators (also available at the JIS), some of which can also display the associated interesting mathematical objects, such as the state diagram. My personal favorite is of course [MA] [GNU]S (written by Matt Levine, Greg Warrington, and me), whereas the author recommends Jack Boyce's excellent `JuggleAnim`; they are both written in Java and can be run from any graphical web browser. As the author himself points out, animators provide motivation and intuition; they will make it easier for the mathematical reader to tackle the heavy combinatorial analysis required for the book's harder theorems.

In case there is any confusion: this is *not* the book to buy in order to first learn to juggle. Indeed, no book can compare to attending a juggling club, where one is guaranteed to find friendly people eager to get one started in juggling. Most colleges and universities have such clubs; there is a list on the JIS with times of (usually weekly) meetings at `http://www.juggling.org/help/meetings`.

Mathematics is a moving target, and there are already results too recent to appear in the book. In [W] is calculated the expectation of finding a juggler in any given state of a state graph, via Stirling numbers. Presumably this tells us something about the affine Weyl group!

References

[CC] *Decisions of the Central Council of Church Bell Ringers*, Morphet, 1989.

[CR] J. Cassidy and B. C. Rimbeaux, *Juggling for the Complete Klutz*, Klutz Inc.

[ER] R. Ehrenborg and M. Readdy, Juggling and applications to q-analogues, *Proceedings of the 6th Conference on Formal Power Series and Algebraic Combinatorics* (New Brunswick, NJ, 1994), *Discrete Math.* **157** (1996), nos. 1–3, 107–125.

[K] A. Knutson, The siteswap FAQ, `http://www.juggling.org/help/siteswap/faq.html`.

[RJ] The USENET newsgroup `rec.juggling`, available through e.g. `http://groups.google.com`.

[W] G. Warrington, Juggling probabilities, `http://front.math.ucdavis.edu/math.PR/0302257`.