Review of *Sync*: The Emerging Science of Spontaneous Order

Reviewed by G. Bard Ermentrout

Sync: The Emerging Science of Spontaneous Order
Steve Strogatz
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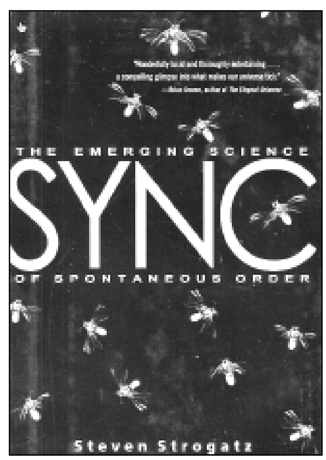
Is the universe a chaotic mess? Or, is it possible that there is much more order than we realize? The obvious answer to the second question is a resounding yes. Otherwise, we would not be here to report on a wonderful new book by Steve Strogatz, *Sync: The Emerging Science of Spontaneous Order*. The recent popularity of books on chaos and complexity might lead their readers to assume that chaos and complexity are the rule. Instead, what Strogatz presents, through his personal scientific experiences, is a contrasting point of view. Interactions between individuals—be they fireflies, pendula, or people—can often lead to the emergence of coherent actions. The goal of this readable and intuitive book is to convince us that this is so through dozens of examples ranging from the spread of rumors to Josephson junctions. The book is autobiographical in tone, and Strogatz plays a central role in much of the science that is described. Part of the fun in reading this book is the sense of excitement that ensues from the mere act of doing science and the discovery of a solution to a long-pondered problem. Phrases like “my hand was sweating as I wrote each new line of the calculation” or “I unleashed the computer and stared at the screen” convey the passionate appeal of mathematical creativity.



Strogatz has condensed and simplified many complex and technical subjects into easily understood stories which, remarkably, are completely equation-free. He achieves this through clever analogies such as runners on a circular track (for synchronization of phase-oscillators) or flushing toilets (for excitable media). This is good for the scientifically literate but mathematically naive reader. For me, it was occasionally a source of frustration, since I wanted more details of the underlying phenomena. Of course, for the interested reader, Strogatz provides thirty-three pages of endnotes which include references, sources, and, importantly, the many caveats. One of the major strengths of the book is the way in which Strogatz finds so many connections between seemingly disparate phenomena. But this is also a weakness, as important differences between the mechanisms underlying these phenomena are sometimes glossed over.

Sync is organized into three areas, each divided into several chapters: sync in living systems (cells, animals, people), sync in inanimate objects (pendula, lasers, electrons), and the “frontiers” of sync (chaos, small worlds, vortex rings). The first three

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chapters of the book cover Strogatz's (and others's) work on pulse-coupled oscillators, globally coupled phase oscillators ("Kuramoto" model), and sleep rhythms. The middle three chapters cover pendula and planets; Josephson junctions and their connections to the Kuramoto model; and my favorite of this section, the instability of the Millennium Bridge in London. The last four chapters describe results on the synchronization of chaotic attractors, Strogatz's and Art Winfree's work on the topology of singular filaments in excitable media, his work with Duncan Watts on "small-world" networks, and finally a summation which makes a few allusions to synchrony and cognition.

In the remainder of this review I will attempt to outline some of the mathematics that is hidden in the pages of this book. I hope to also point out places where the conclusions based on the simple models hold firmly with some generality or fall to pieces with the simplest modifications. The first two sections of the book deal with coupled oscillators in the strict sense; each individual generates a stable periodic solution. At issue is whether there is any collective organization once these individuals are coupled together. Almost all of Strogatz's theoretical work deals with the case of "all-to-all" coupling: the effects of any one unit on another are the same for all possible pairs. In the final section of the book he explores systems which either are not intrinsically periodic or in which the connectivity pattern has additional structure.

The term "sync" is short for "synchrony", by which Strogatz means the emergence of order in time. Depending on whom you are talking to, "synchrony" can have different meanings even within the context of periodic phenomena. In the strictest sense it means that each unit oscillator follows an identical trajectory, $x_j(t) = X(t)$ for all units j. I will call this strict synchrony. Alternatively, "synchrony" is used interchangeably with "phase-locked"; each unit is firing with the same period but there are phase-shifts. Finally, there is a notion of synchrony from statistical physics. Consider

$$X(t) = \frac{1}{N}\sum_j x_j(t)$$

in the limit as $N \to \infty$. If $X(t) = C$ a constant, then the oscillators are said to be asynchronous. The emergence of large temporal fluctuations of $X(t)$ is often defined to be the onset of synchrony. Thus, it is the appearance of temporal order where there was none before. I will make this latter definition more precise when I need it. For systems in which the individuals are, say, chaotic, synchrony is defined as having identical trajectories.

An astonishing example of spontaneous order occurs in Southeast Asia, when thousands of male fireflies congregate in trees and synchronously flash at about once per second. Strogatz begins his book with the following quote from Philip Laurent writing in *Science* in 1917 [1]:

> Some twenty years ago I saw, or thought I saw, a synchronal or simultaneous flashing of fireflies. I could hardly believe my eyes, for such a thing to occur among insects is certainly contrary to all natural laws.

This intriguing phenomena was rigorously addressed by John Buck and his colleagues over a period from the 1930s to the 1980s [2], but the theoretical mechanisms remained unknown. In 1975 Charlie Peskin suggested a simple model for the synchronization of two oscillators, each of which obeys the equation

$$\frac{dx_j}{dt} = I - x_j, \quad I > 1 \tag{1}$$

and such that each time $x_j(t)$ crosses 1 it is reset to 0 and the other oscillator is incremented by an amount ϵ. Peskin analyzed the $N = 2$ oscillator case by explicitly solving the ODEs. Strogatz and his colleague Rennie Mirollo attacked the analogue to this problem for arbitrary N and proved that except for a set of initial data with measure zero, all solutions will synchronize [3]. Thus, they were among the first to rigorously analyze a system of pulse-coupled oscillators. In the introduction of their subsequent paper, they hinted that this global pulse-like coupling provided the answer to the enigma of firefly synchronization. In the discussion section of said paper and here in the endnotes of *Sync*, they explain that the actual mechanism for fireflies is very much different from the simple generalization of the Peskin model.

Instead of restricting their system to the above equation, they assume that each oscillator follows a proscribed temporal profile, $x_j(t) = f(t)$ with $f(0) = 0$ and $f(1) = 1$. (I assume that the period is 1 without loss of generality.) Thus, if an oscillator fires (hits 1), it is reset and all others are given a boost of size ϵ. If the kick is enough to push x_j past 1, then this oscillator is "absorbed" into the pool of synchronized oscillators. (That is, $x_j(t) = \max(1, \epsilon + f(t))$.) We now come to some of the "difficulties" with this model. First, should the effects of m oscillators firing together add $m\epsilon$ to all the others? If the impulse of a firing event pushes an oscillator over 1, then does that instantly increment all others? (In their simulations, they don't allow the oscillators that were pushed over to fire until the next iteration.) The method of proof is very clever and can be understood by considering the case $N = 2$. The idea is to create a map for the phase of oscillator B when oscillator A crosses 1. Let $\phi = g(x_B + \epsilon) \equiv f^{-1}(x_B + \epsilon)$ be the phase of B after A fires. In the time it takes B to fire, A moves to $x_A = f(1 - \phi)$ and then jumps to $\epsilon + x_A$

which has phase $g(\epsilon + f(1-\phi)) \equiv h(\phi)$. Since all we have done is interchange the roles of A and B, the return map is just $R(\phi) = h(h(\phi))$. Strogatz and Mirollo show that there is a unique fixed point and that it is a repellor. To show the latter, they note that

$$h'(\phi) = -\frac{g'(u+\epsilon)}{g'(u)}$$

for some $u \in (0,1)$. They now invoke the first of their hypotheses: $f(t)$ is concave down. This means that $g'' > 0$ so that $h' < -1$ and thus $R' > 1$ for all ϕ. Since there is a unique fixed point and it is a repellor, iterations of the map must go to 0 or 1, which is why the hypothesis of absorption is crucial. Then, with technical difficulties aside, they prove the general N case; any fixed points to the resulting map are unstable. The synchronous state is not really even a fixed point of the map. If one starts with A at 0 and B arbitrarily close to but less than 1, then the phases of A and B are arbitrarily close to each other on the circle. However, they will not stay close; B will fire and A will jump, and they will be nearly ϵ apart until A fires again. Small changes in the model such as heterogeneity mean that the system can synchronize only in the sense of absorption. Since synchrony is not a fixed point of the system, there is no notion of local stability, so that small changes in the model can have dramatic effects. If the coupling is not "all-to-all", another problem occurs: not all oscillators will be advanced equally when there is a firing of some group. For example, with nearest neighbors, is synchrony inevitable in this model?

How does this simple model compare to real fireflies or other systems coupled in a pulsatile fashion? Little is known about the transduction of the visual signal to the alteration of the oscillator. However, it is possible to quantify this effect by looking at phase transition curves (PTC). If a pulse stimulus is given to an oscillator at phase ϕ, then the phase will often be shifted to a new phase $F(\phi)$; this function is called the phase transition curve. Strogatz's mentor, Art Winfree, tabulated and measured such curves for a variety of biological oscillators. Frank Hanson (who worked with Buck) measured $F(\phi)$ for several firefly species [4]. In the Strogatz-Mirollo (SM) model, the phase is *always advanced* from a stimulus no matter what the phase. Furthermore, $F(\phi)$ is not one-to-one, since a nonzero interval of phases is mapped to 1. However, in the insect *Pteroptyx malaccae* the PTC is qualitatively like

$$F(\phi) = \phi - \epsilon \sin 2\pi\phi$$

where ϵ is a small positive number. This PTC obviates some of the difficulties that are inherent in the SM model. As long as ϵ is small enough, F is monotone, so no single stimulus can ever make the oscillator fire instantly. Furthermore, the analogous return map for this PTC has 0 as an asymptotically stable fixed point. In the all-to-all case, it can also be shown that synchrony is asymptotically stable for this PTC [5]. Whether sync is inevitable in the all-to-all coupled case with the sinusoidal PTC remains an open mathematical question.

In my opinion, Strogatz's best work is his incisive analysis of the Kuramoto model described in Chapter 2. (This was the model which led to his aforementioned sweaty palms.) The Kuramoto model is a simple coupled system of phase oscillators:

$$\frac{d\theta_i}{dt} = \omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i). \tag{2}$$

Such models arise through averaging of systems of weakly coupled nonlinear oscillators. Depending on the specifics of the model, the function $\sin(\theta)$ is replaced by an arbitrary periodic function. In a later chapter in *Sync*, Strogatz shows that the Kuramoto model is exact when the physical system is an array of Josephson junctions. When Kuramoto first proposed the model, he considered it to be a toy model for the interactions of biological oscillators. All-to-all coupling has many advantages over interactions with a more specific topology, where details of the boundary conditions and heterogeneities make a general analysis impossible. Kuramoto [6] was interested in the behavior as a function of the coupling parameter K when ω_i are drawn from a symmetric probability density function $g(\omega)$ with zero mean. Numerical solutions for large N revealed that as K increased, there was a transition from complete disorder to order. Key to his subsequent analysis was the fact that the sum in (2) can be written as

$$-R\sin(\theta_i) + Q\cos(\theta_i)$$

where

$$R = \frac{1}{N}\sum_j \cos\theta_j, \qquad Q = \frac{1}{N}\sum_j \sin\theta_j.$$

Kuramoto argued that since the frequency distribution was symmetric about 0, then Q should be zero, for then the equations are symmetric under the transformation $\theta \to -\theta$. Since the mean frequency of the oscillators is zero, he needed to solve:

$$0 = \omega_i - KR\sin\theta_i \tag{3}$$

along with the self-consistency condition

$$R = \frac{1}{N}\sum_j \cos\theta_j. \tag{4}$$

Kuramoto's insight was to divide the oscillators into two groups: those for which equation (3) has a solution ($|\omega| < KR$) and those for which it does not. The latter oscillators drift around the circle, and Kuramoto intuited that their contribution to (4) should be zero over long time. The sum is just the average of $\cos\theta$ over the frequency distribution defined by $g(\omega)$, so that we can replace it by the integral

$$R = \int_{-KR}^{KR} \cos(\theta(\omega)) g(\omega) d\omega$$

where $\sin(\theta(\omega)) = \omega/KR$. A change of variables leads to the following equation for R:

$$R = KR \int_{-1}^{1} g(KR\sigma)\sqrt{1-\sigma^2}\, d\sigma. \tag{5}$$

Clearly, $R = 0$ is always a solution. However, a nonzero solution bifurcates at the critical value

$$K_c = \frac{2}{\pi g(0)}.$$

This is a clever argument, but it is completely heuristic. Strogatz relates how he and Nancy Kopell worked on trying to make this rigorous with no results and how he was plagued by it for many years. He then describes in breathless prose how it came to him in a near-dream state: the oscillators are not runners on a track, but like a fluid. Thus, he was led to write an equation for the *density* of oscillators at phase θ and with natural frequency ω. The density evolves as

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\nu\rho)}{\partial \theta} \tag{6}$$

where ν is the phase velocity given by

$$\nu(\theta, t) = \omega + K\int_{-\infty}^{\infty} g(\sigma) d\sigma \times \int_0^{2\pi} d\phi \sin(\phi - \theta)\rho(\phi, \sigma, t).$$

This approach has many advantages over the Kuramoto approach. It does not depend on the fact that the interaction is sinusoidal. Furthermore, additive noise in the original model simply adds an additional flux term, $D\partial^2\rho/\partial\theta^2$, to the density equation. The key point is to notice that the asynchronous state is the state in which the phases are uniformly distributed around the circle: $\rho(\theta, \omega, t) = 1/2\pi$. Clearly, this is a stationary state for (6). Thus, the question that Strogatz asked was, how does the stability depend on the strength of the coupling K? Linearizing about this stationary solution leads to an eigenvalue problem whose solution is

$$y(\theta, \omega, t) = e^{\lambda t} e^{i\theta} b(\omega)$$

and

$$\lambda b(\omega) = -i\omega + \frac{K}{2}\int_{-\infty}^{\infty} g(\omega) b(\omega)\, d\omega.$$

Solving this for the discrete eigenvalue λ yields

$$1 = \frac{K}{2}\int_{-\infty}^{\infty} \frac{\lambda}{\lambda^2 + \omega^2} g(\omega)\, d\omega.$$

Rescaling $\omega = \lambda\sigma$ results in

$$1 = \frac{K}{2}\int_{-\infty}^{\infty} \frac{g(\lambda\sigma)}{1+\sigma^2}\, d\sigma.$$

Bifurcation occurs in the limit as $\lambda \to 0$, yielding the Kuramoto result:

$$1 = \frac{\pi K_c}{2} g(0).$$

There are many subtleties to this full eigenvalue problem (the essential spectrum), and the reader is urged to consult the lovely review article [7]. But the bottom line is that the phase density approach finally resulted in a rigorous solution to Kuramoto's model. (I should point out that the large but finite N case is still the subject of current research.) This approach remains the method of choice when trying to understand the stability of the asynchronous state. This elegant calculation is the mathematics which underlies Strogatz's statement that "sync" is inevitable. How general is this statement? J. D. Crawford answered this question in a series of papers on the model

$$\frac{d\theta_j}{dt} = \omega_j + \frac{K}{N}\sum_k H(\theta_k - \theta_j)$$

where H is a more general interaction function. For example, if $H(\theta) = \sum_n c_n \sin n\theta$, then for each n there is a critical coupling strength,

$$K_n = \frac{2n}{g(0)\pi c_n}.$$

If K_n is the minimal for $n = n_0$, then the probability distribution which bifurcates will have n_0 peaks. The Kuramoto case corresponds to $n_0 = 1$ and so there is "sync". However, if, e.g., $n_0 = 2$, then a two-cluster state will bifurcate. Thus, "sync" is not really inevitable; it was an accident of the model choice.

The daily 24-hour rhythms which govern our everyday behavior and which are the subject of Chapter 3 are controlled by a small set of neurons buried deep in a part of the brain called the suprachiasmatic nucleus (SCN). In the last twenty years, biologists have uncovered the mechanisms of the genesis of these rhythms. However, the coupling between them remains mysterious. It has been suggested that coupling is through the release of a chemical transmitter (gamma aminobutyric acid, or GABA) that has been shown to affect the phase of individual oscillators. It is possible that this transmitter is pooled so that the cells are effectively coupled in an all-to-all manner. But coupled oscillators play little role in this chapter of *Sync*. Rather, Strogatz focuses on the rather extreme experiments on human subjects kept in isolation from all time cues. In early experiments, volunteers were also isolated from other people and exhibited a variety of psychological problems. Most interestingly, after a long period with no temporal cues, the sleep-wake cycle "separates" from the daily body temperature fluctuations that are controlled by the SCN. This spontaneous desynchronization is what captured the attention of theoreticians. The main idea of the theory is that there is a sleep-wake oscillator and a circadian oscillator that are coupled together but have different intrinsic frequencies. Both are driven by the day-night cycle that keeps them entrained at a 24-hour rhythm. In the absence of the temporal cues, the coupling between the two oscillators is not enough to maintain locking, and they drift apart. Strogatz spends a majority of the chapter describing his work with Dick Kronauer, a modeler of circadian rhythms, and how their data revealed that during certain times of the day it is very difficult to stay awake (around 2:00 p.m., siesta time in many reasonable societies) or go to sleep (around 10:00 p.m., dinner time in those same societies).

The next three chapters deal with physical systems; the most notable (at least to me) is Huygens's clocks. This is a classic example of synchrony in which two pendulum clocks mounted on a beam phase-lock so that they operate a half a cycle out of phase. Strogatz refers to a recent paper by Matthew Bennett and others [8] in which the mystery of this locking is finally solved. Bennett's model is two pendula attached to a rigid beam that is allowed to move vertically. This movement is crucial, since without it the two pendula could not be coupled. It is easy to write down a Lagrangian for this three degree-of-freedom model and solve it numerically. By a simple linear approximation, they reduce the behavior to a map. With this map, Bennett et al. show that Huygens was incredibly lucky: had his pendula been heavier, they would have been too weakly coupled to phase lock; and had they been much lighter, then a solution in which one pendulum rocks and the other is silenced would have occurred. Strogatz uses the example of Huygens's clocks as a perfect illustration of how inanimate objects can sync. The remainder of Chapter 4 gives more examples, such as the electrical grid and the famous Kirkwood gaps in the asteroid belt between Mars and Jupiter. Kirkwood noticed that these occur at radii in which the orbital period is resonant with that of Jupiter. It took another hundred years before a theory was developed that showed this resonance could produce gaps [9].

Chapter 5 describes, mostly through analogies, sync at the quantum level. Strogatz has a nice description of how lasers work, an example of quantum phase coherence. The idea is that occasional photons will join up in sync, and these will recruit more and more of them, and thus there is a positive feedback to produce phase-coherent light. He closes this chapter with a biography of one of the most interesting characters in quantum mechanics, Brian Josephson. The final chapter in this part of the book contains the majority of Strogatz's contributions to "inanimate sync". He makes connections between quantum sync and the Kuramoto model through his prolific work on coupled Josephson junctions which obey the following differential equations:

$$\begin{aligned} \beta_j\phi_j'' + \phi_j' + \sin\phi_j + Q' &= I_j \\ LQ'' + RQ' + C^{-1}Q &= \frac{1}{N}\sum_k \phi_k'. \end{aligned}$$

He first describes observations on the existence of an invariant 2-torus when $C^{-1} = L = 0$. This is a difficult concept to explain to mathematicians, let alone a general audience. But he neatly trumps this difficulty by describing the torus as a sequence of nested Russian dolls. In the limit of small Q' ("weak coupling"), it is possible to average the junction equations, and this is how he ends up with Kuramoto's original model of sinusoidal all-to-all coupling. Thus, while it may be hard to find a biological system that reduces to the exact form of Kuramoto's famous equations, Strogatz has shown that at least one physical system is equivalent.

The middle section of the book concludes with a great description of the Millennium Bridge in London. This bridge was built to celebrate the new millennium, and at its inaugural with all the television cameras running, hundreds of people began to walk across it. As Strogatz notes, "Within minutes it began to wobble, 690 tons of steel and aluminum swaying in a lateral S-shaped vibration like a snake slithering on the ground." He presents a mechanism based on a combination of resonance and phase-locking. This is not like the famous Tacoma Narrows Bridge, which broke apart when wind of a critical velocity blew across it. Rather, when people walk they tend to swing from side to side at roughly two strides per second. Engineers found that the bridge itself had a resonant instability of about one cycle per second. If through purely random chance a number of people transiently synchronized their gaits, this net force could begin to destabilize the bridge so that it starts rocking slightly. If you try to walk on a swaying platform, you tend to compensate by stepping in sync with it. Thus this initially small kernel of synchronous steppers recruited more and more walkers, which further destabilized the bridge, leading to the moral equivalent of the Kuramoto model! As a last word on inanimate sync, Strogatz notes (with an implied wink) that three days after the fiasco (and several months before the engineers analyzed the problem), a letter appeared in *The Guardian* giving essentially the above analysis of the phenomena. The author: Brian Josephson.

Part three of the book covers what is called the "frontiers of sync". I am not exactly sure what this means; Strogatz's work on singular filaments in excitable media (Chapter 8) was done nearly twenty years ago. Nevertheless, the other chapters in this section describe more recent work on chaotic synchronization, small worlds, and a very small bit on neuronal synchrony.

Lou Pecora's work occupies a good part of the text in Chapter 7. He has written a number of papers on synchronization of chaotic systems [10]. Chaotic synchronization asks the simple question as to whether two or more coupled chaotic systems will stably synchronize. Let me start with a very general model that will work for periodic oscillations as well as synchronous rhythms. Consider:

$$\frac{dX_j}{dt} = F(X_j) + \sum_k c_{jk} M(X_k, X_j), \quad j = 1, \ldots, N,$$

where $X \in R^m$ and $M(X, X) = 0$. The coupling strength is encoded in the scalars c_{jk}. Suppose that $X' = F(X)$ has a solution $U(t)$ that may be chaotic or periodic but which is time-varying. Clearly $X_j(t) = U(t)$ is a synchronous solution. Stability is determined from the linearized equation $X_j = U + Y_j$ where

$$\frac{dY_j}{dt} = A(t)Y_j + \sum_k d_{jk} B(t) Y_k.$$

where A, B are time-varying $m \times m$ matrices and d_{jk} is the same as c_{jk} except for $j = k$ where $d_{jj} = c_{jj} - \sum_k c_{jk}$. Let (ν, Ψ) be an eigenvalue-eigenvector pair for the $N \times N$ matrix $D = (d_{jk})$. Write $Y_j = \Psi_j Z$ where $Z \in R^m$ and Ψ_j is the j^{th} component of Ψ. This has the effect of putting the system into block diagonal form, leaving us with only the set of problems:

$$\frac{dY}{dt} = A(t)Y(t) + \nu B(t)Y(t) \tag{7}$$

where ν is an eigenvalue of D. Suppose the solution $U(t)$ is chaotic. Then stability is determined by looking at the maximal Liapunov exponents of (7). Since 0 is an eigenvalue of D corresponding to homogeneous perturbations, this means that for a chaotic system there is always at least one positive Liapunov exponent. But, *it is in the homogeneous eigenspace*; thus this instability cannot break the symmetry of synchrony. Pecora's idea for chaotic synchronization is very easy. Find regions in the complex ν-plane for which the Liapunov exponents are all negative. This will characterize the types of coupling that can stabilize the synchronous chaotic state. Suppose, for example, $B(t) = bI$ where b is a scalar and the maximal Liapunov exponent of the uncoupled system is λ. If the nonzero eigenvalues of D are negative, then for b large enough, the synchronous system is stable. If $B(t)$ is a constant nonscalar matrix, then there can be limitations on the strength of the coupling in order to get chaotic synchronization. The methods described here work as well for coupled oscillators. Stability of the synchronous chaotic state is an exercise in linear algebra coupled with numerics.

In the first two-thirds of *Sync* the interactions between individuals is "all-to-all", so that the behavior of the ensemble is pretty simple: there is synchrony or asynchrony, but, of course, no spatial structure. Not until Chapter 8, "Sync in Three Dimensions", does Strogatz break away from this simplifying constraint. Here he first describes how he came to know one of his heroes, Art Winfree, who tragically died last year and to whom the book is dedicated. Winfree was studying active media in the form of the Belousov-Zhabotinsky (BZ) reagent, a mixture of chemicals which generates spontaneous oscillations and spiral waves in a thin layer. (When I first met Winfree at a conference, he passed out specially treated millipore filters and small vials of clear liquid. When you poured the liquid on the filter paper, the BZ reaction took place and spiral waves appeared miraculously on the paper.) Winfree was interested in what could happen in three dimensions and recruited Strogatz to work with him one summer. Imagine a stack of spiral waves exactly lined up; the centers or cores of the spirals produce a straight line (called a singular filament). This produces a scroll wave. Now suppose that we could take such a filament and join the ends to form a ring. The resulting object is called a scroll ring, and associated with each point on the ring is a two-dimensional spiral wave. It is easy to imagine more complex objects such as a trefoil knotted ring or linked rings. In a series of papers published between 1983 and 1984, Winfree and Strogatz [11], [12] described the kinds of patterns that are consistent with chemical and biological excitable media. First, consider a simple planar spiral wave. The isophase lines of this spiral converge in a phase-singularity at the center of the spiral wave. Thus, our filament represents a curve of phase singularities. Consider the simple twisted scroll ring which is obtained as follows. Take a scroll (spiral waves stacked up with the arms aligned) and give it a 360-degree twist and then join the ends. Now imagine a torus that encloses the twisted ring. A surface of the wave at a fixed phase intersects the torus in a closed ring. At the equator of the torus, the isophase points wind around exactly once. Now plug the hole in the torus with a disk; the isophase lines must come together at *another* singularity. Thus the twisted scroll ring cannot exist in isolation. Using clever arguments like this, Winfree and Strogatz show that more exotic possibilities are realizable: for example, a pair of linked twisted scroll rings. The above-mentioned trefoil knot *untwisted* cannot exist; however, with a twist it is realizable. In later work (not with Strogatz) Winfree simulated a simple excitable medium and found apparently stable (or at least long-lasting) knotted twisted scroll rings.

In my mind this chapter is not so well connected to the rest of the book. Let me try to smooth the seams a bit by returning to the coupled phase-models that play a role in Chapter 2 of *Sync*:

$$\frac{d\theta_j}{dt} = \omega_j + \sum_k c_{jk} H(\theta_k - \theta_j), \qquad j = 1, \ldots, N. \tag{8}$$

The Kuramoto model is a special case of this general coupling, where $H(\theta) = \sin\theta$ and $c_{jk} = 1/N$. We have already seen that for the all-to-all case the Kuramoto story is more complex if the coupling is not a pure sinusoid. The zoo of exotic waves described in this chapter is far more complex than sync. Sync for these media, which are essentially homogeneous, would represent either a stable resting state or a bulk oscillation. For chemical and other media these simple states are stable. So where do these exotic creatures originate? The answer lies in the interactions in the medium. Unlike all of the previous models in *Sync*, interactions between individual elements are local in space; coupling is not "all-to-all". Stirring the chemical bath destroys any hope of finding spirals and scroll waves. So, let's make the question really simple. Is sync inevitable in equation (8) when coupling is local and the medium is homogeneous? Consider first a ring of nearest neighbor sinusoidal phase oscillators:

$$\frac{d\theta_j}{dt} = \omega + \sin(\theta_{j+1} - \theta_j) + \sin(\theta_{j-1} - \theta_j), \qquad j = 1, \ldots, N.$$

We identify $j = 0$ with $j = N$ and $j = N + 1$ with $j = 1$. One possible solution to this model is a rotating wave, $\theta_j = \omega t + 2\pi j/N$. It is asymptotically stable as long as $N > 4$. However, if we increase the range of the coupling sufficiently far beyond the nearest neighbor coupling, all that can occur is sync. One might object that the ring is special compared to a simple linear array. Indeed, for a linear nearest-neighbor array of sine models, the following can be proved: suppose that $-\pi/2 < \theta_{j+1}(0) - \theta_j(0) < \pi/2$. Then all solutions go to synchrony. So, based on this, one could conjecture that for *nearest-neighbor linear arrays of sine oscillators*, synchrony is inevitable.

However, what may be surprising is the behavior when we go to two dimensions. Sync is not inevitable for a nearest-neighbor array of sine oscillators on a square grid:

$$\frac{\theta_{ij}}{dt} = \omega + \sum_{(i',j') \in N(i,j)} \sin(\theta_{i'j'} - \theta_{ij}),$$

where $N(i,j)$ is the set of nearest neighbors of (i,j). Thus $(1,1)$ only has two neighbors, $(1,2)$ and $(2,1)$. For a nearest-neighbor sine model on an $2m \times 2m$ grid ($m > 1$), there are stable rotating patterns that are the discrete analogues of spiral waves in the two-dimensional excitable medium [13]. For m small the basin of attraction of these oscillating but nonsynchronous patterns is small, but for m large the basin for synchrony shrinks. Thus spirals and other patterns are consequences of having a well-defined notion of space such as in locally coupled media. Stacking these sine-model rotating spirals up in a three-dimensional array results in the analogue of a simple scroll wave. Whether exotic structures like scroll rings exist in three-dimensional arrays of locally coupled oscillators remains an open question.

The penultimate chapter in *Sync* covers Strogatz's work with Duncan Watts on the "six degrees of separation" problem. There is a famous game called the "Kevin Bacon" game that started in a Pennsylvania college. The idea is to pick any Hollywood actor and connect him to Kevin Bacon in as few steps as possible. For example, to connect Kevin Bacon to Woody Allen, we note that Bacon was in *Footloose* with Dianne Wiest, who was in *Hannah and Her Sisters* with Woody Allen. Mathematicians play this game with Paul Erdős. You can imagine a network with each actor as a node and links drawn between each actor that appeared in a movie together. Such a network is somewhere between a completely local one and one which is completely random. Many realistic networks have a similar structure to the Erdős and Bacon networks, such as the Internet, the national power grid, the neurons of the nematode *C. elegans*, and many social networks. There are many local links and a few nodes that have links to many other nodes, some quite distant. This type of network is called a "small world network". The material in this chapter is mainly taken from Watts's Ph.D. thesis and subsequent book [15], reviewed in the *Notices* [14]. I should point out that Barabási and his collaborators have developed an alternate theory for scale-free networks [16].

The final chapter is a miscellany of results on the "human side of sync", covering such things as the spread of fads and an interesting phenomena regarding hand clapping. In Eastern Europe, after a good concert the audience often claps rhythmically. This rhythmic clapping can suddenly become asynchronous and then switch back to synchrony. This was recently quantified in [17] by looking at the maximal correlation between a ceiling recording of the audience, $c(t)$, and a sinusoidal function of t. The mechanism is rather interesting. In order to synchronize, the audience must slow down their clapping. But this results in a diminution of the sound intensity of the clapping, contradicting their desire to make the applause as loud as possible. Thus, they speed up and lose synchrony. This leads to a waxing and waning of the synchronous behavior. At high frequencies the spread of individual clapping rates is much greater than at low frequencies, so that the authors suggest that the onset of sync at low frequencies is through the Kuramoto mechanism. There is another possible reason for the loss of synchrony: coupled neural oscillators can often switch from synchrony to asynchrony as the frequency changes, even when the oscillators have identical frequencies [18], [19].

This leads me to the last part of *Sync*, in which Strogatz briefly dives into the neural synchrony tar pit. He touches on work by Charlie Gray, Wolf Singer, and others in which it was suggested that synchronous oscillations may have some importance in cognition; this remains a major controversy in the field. He also cites the famous story in which hundreds of Japanese got seizures from watching an episode of *Pokemon* that featured flashing red lights at a particularly sensitive frequency. The highlight of this chapter, however, is his very funny story about a lunch with Alan Alda in the MIT lunchroom.

Like his textbook on nonlinear dynamics, the present book is a model of clarity. Reading *Sync* has been fun and intellectually stimulating. I can think of about a dozen mathematical questions that came to mind while reading different parts of the book. While it is written for the layperson, there is plenty of interesting mathematics behind the scenes; the endnotes provide many references. I hope that *Sync* will be read by many nonmathematicians so that they can appreciate both the usefulness of mathematics and other forms of "pure" science. And the next time someone asks what I do for a living, I cannot think of a better place to point her.

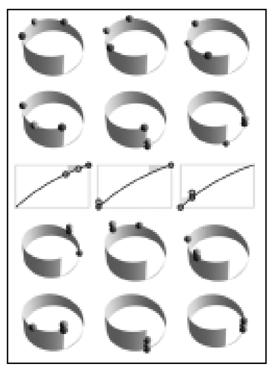
References

[1] Philip Laurent (1917), *Science* **45**:44.
[2] John Buck (1988), *Quart. Rev. Biology* **63**:265–289.
[3] Renato Mirollo and Steve Strogatz (1990), *SIAM J. Appl. Math.* **50**:1645–1662.
[4] F. E. Hanson (1978), *Fed. Proc.* **37**:2158–2164.
[5] P. Goel and B. Ermentrout (2002), *Phys. D* **163**:191–216.
[6] Y. Kuramoto (1984), *Chemical Oscillations, Waves, and Turbulence*, Springer-Verlag, Berlin.
[7] Steve Strogatz (2000), *Phys. D* **143**:1–20.
[8] M. Bennett, M. Schatz, H. Rockwood, and K. Wiesenfeld (2002), *Proc. Royal Soc. London* **458**:563–579.
[9] N. Murray and M. Holman (2001), *Nature* **410**:773–779.
[10] L. M. Pecora and T. L. Carroll (1990), *Phys. Rev. Lett.* **64**:821–824.
[11] A. T. Winfree and S. Strogatz (1983), *Phys. D* **9**:333–345
[12] ____ (1984), *Phys. D* **13**:221–233.
[13] J. Paullet and B. Ermentrout (1994), *SIAM J. Appl. Math.* **54**:1720–1744.
[14] Carson Chow (2000), *Notices Amer. Math. Soc.* **47**(8):880–883.
[15] Duncan Watts (1999), *Small Worlds*, Princeton University Press, Princeton, NJ.
[16] Albert-László Barabási (2002), *Linked: The New Science of Networks,* Perseus, NY.
[17] Z. Neda et al. (2000), *Phys. Rev. E* **61**:6987–6992.
[18] C. van Vreeswijk, L. Abbott, and B. Ermentrout (1994), *J. Comput. Neurosci.* **1**:313–321.
[19] C. C. Chow and N. Kopell (2000), *Neural Comp.* **12**:1643–1678.

About the Cover

Synchronizing Peskin's heart

The cover this month illustrates how beat synchronization occurs in a heart model proposed around 1975 by Charles Peskin (in the Courant Institute notes "Mathematical aspects of heart physiology").



The heart contains a large number of virtually identical cells that must fire more or less simultaneously (in a healthy heart), and if disturbed, they must return to the synchronized state. In Peskin's model the state of each cell is characterized by its phase, and the system itself is also characterized by a function of phase whose graph is concave (as in the middle row of the cover). At the end of every cycle, each cell fires and adjusts the phase of every other cell, translating it into synchronization with itself or adjusting its phase so as to raise the value of the function by a fixed amount.

Renato Mirollo and Steven Strogatz were the first to prove that Peskin's model incorporates synchronization of an arbitrary population (*SIAM Journal of Applied Mathematics* **50** (1990)). An informal explanation is given in the book by Strogatz reviewed here.

The cover shows how this process works with three cells. It demonstrates roughly how concavity of the graph plays a role.

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