Abel's Proof: An Essay on the Sources and Meaning of Mathematical Unsolvability

Reviewed by Lars Gårding

Abel's Proof: An Essay on the Sources and Meaning of Mathematical Unsolvability
Peter Pesic
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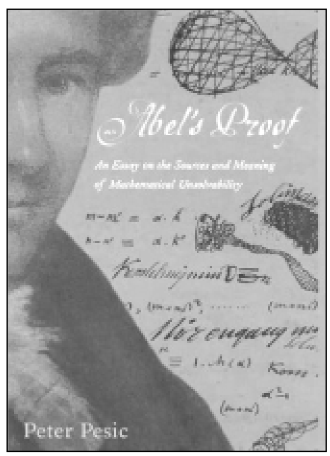
The general polynomial equation of degree 2 was solved in antiquity. Much later, around 1500, it was discovered in Italy that the zeros of general equations of degrees 3 and 4 can be expressed in terms of the coefficients by repeated rational operations and root extractions. One difficulty of this method in more general cases is that an nth root has n values so that simple expressions obtained by rational operations and root extractions from the coefficients easily take on more values than there are zeros of the equation. To give simple examples, the third root of $2+\sqrt{3}$ has six values, and adding to it the third root of $3+\sqrt{2}$ gives a root expression with thirty-six values. It is only artful combinations of similar expressions that produce the zeros of the equations of degree less than 5 and nothing else. For instance, in the formula for the zeros of the general equation of degree 3, two third roots are added whose product is required to be one of the coefficients. In spite of the many values of complex root expressions, it turned out in the end that they are still not flexible enough to represent the five zeros of the quintic, i.e., the general equation of degree five. But in the beginning of the nineteenth century past successes made it perhaps natural to be optimistic about the equation-solving power of roots or, with their Latin name, radicals. It may even have appeared that the only way of solving equations was by radicals.



That the quintic is not solvable by repeated root extractions and rational operations on the coefficients is in fact one of the most widely known results in mathematics. The discovery has a long prehistory and a dramatic finale in the first half of the nineteenth century with two heroes, Abel and Galois. Both died in their twenties, and their fame came afterwards. Abel was first with a proof that the quintic is not solvable by repeated rational operations and root extractions starting with the coefficients, and very soon both Abel and Galois gave what is essentially the end result, a beautiful theory of equations in general and solvable equations in particular.

The story of this development has been told in many ways: as history, as the lives of young geniuses; and as the terse, five-page account of the entire theory in van der Waerden's *Algebra*.

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Peter Pesic's book, entitled simply *Abel's Proof*, turns it into a broad, lively, and rather personal account of Abel's proof and its prehistory. The author even uses three appendices to give accounts of Abel's proof and other related mathematical truths: for instance, that the alternating group of permutations of more than four elements has no invariant subgroups. Here his essayistic style and frequent appeals to basics makes his text very difficult for anyone trying to achieve a solid grasp of the material. But this is just a mathematician's remark. The ordinary reader looks for readability rather than a text requiring tough thinking.

The first eight chapters give a fairly straightforward account of the prehistory of unsolvability in radicals of the general quintic. Under the heading "The Scandal of the Irrational", a reader starts the journey in antiquity with Euclid and the questionable existence of irrationals. In the next chapter, "Controversy and Coefficients", the reader passes on to the fifteenth and sixteenth centuries with emerging algebra and equations and the solutions of the equations of the third and fourth degrees by Ferrari and Cardano. In the chapter "Impossibilities and Imaginaries" the reader meets the imaginary numbers; and in the next one, with the enigmatic title of "Spirals and Seashores", he is introduced to preludes to the quintic, including Gauss's proof that every equation has a root—in the reviewer's mind perhaps a more important result than the unsolvability of the quintic.

After yet a chapter on preludes to the quintic, "Premonitions and Permutations", that includes Lagrange's resolvents, the reader reaches a nine-page chapter called "Abel's Proof". It is followed by a historical comparison between the works of Abel and Galois.

When the mathematics gets serious, as in the chapter on Abel's proof, the author is soft on the reader, separating a number of fact boxes of formulas from his running account. In the long-winded fourteen pages of Appendix A, Abel's proof is treated as an object to be seen from all sides. The author's choice to make Abel's proof the high point of the story is historically motivated, but I think that a reader would have been better informed by appendices also on what followed afterwards, i.e., the theory of equations solvable by rational operations and root extractions beginning with the coefficients. Anyway, trying to read Appendix A made the reviewer long for an arrangement that avoids Abel's somewhat awkward proof. This could be done in an appendix explaining and exemplifying the current theory of solvability by radicals in the usual terminology. The climax would then be a brief and lucid argument showing why a solvable quintic cannot be general.

After the preceding chapters the historical theme that kept the book together gives way to more loosely connected, general material. In "Seeing Symmetries" the author sets about to combine explanations of symmetry groups with those of the classical regular bodies where mappings are illustrated by dancers changing places. In the mind of the reviewer this approach creates more confusion than order in a reader wanting to get an idea of what a group is. Like many other places in the book, the trouble here is a style that prevents the author from presenting tight (but highly informative) material.

Of the two last chapters, "Seeing the Order of Things" and "Solving the Unsolvable", the first one deals with the general implications of noncommutativity, including quantum mechanics and relativity theory. But here the line of reasoning ranges over centuries, and a multitude of noncommittal remarks and predictions makes this chapter more obscure than interesting. The implications of Abel's and Galois's result are extended to many cases having only a slender connection with their supposed origin, as when trying to sum up Abel's importance: "The end of the old assumption that all equations have a finite solution revealed a new mathematics of infinite series and noncommutativity." The same is true of the last chapter, which among other things tries to attach some higher philosophical sense, or perhaps, just a pun, to the notion of unsolvability: "By transcending this limitation (unsolvability by radicals) Abel solved the unsolvable."

However, these two quotes do not give a fair idea of the book. The author has written a good historical account of how to solve and not to solve equations by radical extraction that will give a future reader many hours of pleasure and general historical knowledge. I should also add that the book has a very extensive, useful, and reasoned list of references.