The Status of the Classification of the Finite Simple Groups

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Common wisdom has it that the theorem classifying the finite simple groups was proved around 1980. However, the proof of the Classification is not an ordinary proof because of its length and complexity, and even in the eighties it was a bit controversial. Soon after the theorem was established, Gorenstein, Lyons, and Solomon (GLS) launched a program to simplify large parts of the proof and, perhaps of more importance, to write it down clearly and carefully in one place, appealing only to a few elementary texts on finite and algebraic groups and supplying proofs of any "well-known" results used in the original proof, since such proofs were scattered throughout the literature or, worse, did not even appear in the literature. However, the GLS program is not yet complete, and over the last twenty years gaps have been discovered in the original proof of the Classification. Most of these gaps were quickly eliminated, but one presented serious difficulties. The serious gap has recently been closed, so it is perhaps a good time to review the status of the Classification. I will begin slowly with an introduction to the problem and with some motivation.

Recall that a group G is *simple* if 1 and G are the only normal subgroups of G; equivalently $G \cong G/1$ and $1 \cong G/G$ are the only factor groups of G.

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Assume G is finite, and write $H \trianglelefteq G$ to indicate that H is a normal subgroup of G. A *normal series* for G is a sequence

$$1 = G_0 \triangleleft G_1 \triangleleft \cdots \triangleleft G_n = G,$$

and the family of *factors* of the series is the family

$$(G_{i+1}/G_i : 0 \leq i < n)$$

of factor groups arising in the sequence. Observe the series is maximal (i.e. we cannot adjoin an extra term between G_i and G_{i+1}) precisely when each factor is simple. The maximal series are called *composition series*, and we have:

Jordan-Hölder Theorem. *All composition series for G have the same length and the same (unordered) family of simple factors.*

The simple factors in a composition series for G are called the *composition factors* of the group. They do not determine G up to isomorphism, but they do exert a lot of control over the gross structure of G. Thus the simple groups are analogous to the primes in number theory, although one does not quite have "unique factorization".

Example. Recall G is *solvable* if each of its composition factors is of prime order. Recall also that Galois's Theorem says this is the class of groups corresponding to polynomials solvable by radicals. Further, by a result of Philip Hall, solvable groups satisfy an important generalization of Sylow's Theorem, and indeed this property characterizes solvable groups. This generalization says: If G is solvable of order n and m is a divisor of n with

$(m, n/m) = 1$, then G has a subgroup of order m, all subgroups of G of order m are conjugate in G, and each subgroup of order dividing m is contained in some subgroup of order m.

One can conceive of an analysis of the finite groups based on a solution to the following two problems:

The Classification Problem. *Determine all finite simple groups.*

The Extension Problem. *Given groups X and Y, determine all extensions of X by Y; i.e. determine all groups G with a normal subgroup H such that $H \cong X$ and $G/H \cong Y$.*

In practice the Extension Problem is too hard, except in special cases. It seems better not to look too closely at the general finite group, but instead when faced with a problem about finite groups, to attempt to reduce the problem or a related problem to a question about simple groups or groups closely related to simple groups. Then using the Classification of the finite simple groups and knowledge of the simple groups, solve the reduced problem. Note this procedure works only if one knows enough about simple groups to solve the problem for simple groups; this is where the Classification comes in: it supplies an explicit list of groups which can be studied in detail using the effective description of the groups supplied by the Classification.

This approach to solving group theoretic problems has been in use since about 1980, when the finite simple groups were deemed to have been classified. It has been extremely successful: virtually none of the major problems in finite group theory that were open before 1980 remain open today. Moreover, finite group theory has been used to solve problems in many branches of mathematics.

In short, the Classification is the most important result in finite group theory, and it has become increasingly important in other areas of mathematics.

Now it is time to state the:

Classification Theorem. *Each finite simple group is isomorphic to one of the following groups:*

1. A group of prime order.

2. An alternating group.

3. A group of Lie type.

4. One of 26 sporadic groups.

Observe that the statement of the Classification Theorem given above is deceptively simple. In order for it to have real content, one must define what one means by "group of Lie type" and "sporadic group". Constraints on space preclude including such definitions here, so instead I will make the following observation: With the exception of some of the sporadic groups, each group G appearing in the theorem can be regarded as essentially the group of automorphisms of some fairly accessible mathematical object X. The existence of X gives an existence proof for G, but, more important, the representation of G on X gives a means for studying G and obtaining information about representations of the simple groups in various categories, most particularly representations as permutation groups (i.e. subgroup structure of G) and linear groups. The verification of some special property of such representations is usually the sort of extra input necessary to apply the Classification to solve a given problem.

Indeed some such information was necessary to prove the Classification in the first place. That is to say, in proving the Classification Theorem, one begins with the list $\mathcal{K}$ of simple groups appearing in the statement of the theorem and considers a minimal counterexample to the Classification Theorem: A finite simple group G minimal subject to $G \notin \mathcal{K}$. Thus each proper subgroup J of G is a *$\mathcal{K}$-group*: if $K \trianglelefteq H \leq J$ with H/K simple, then the *section* H/K is in $\mathcal{K}$. The proof of the Classification depends very heavily on facts about the subgroup structure and linear representations of $\mathcal{K}$-groups.

During this article I will be discussing two different efforts: The "original proof" of the Classification and work done since about 1980 aimed at improving, simplifying, and writing down carefully in one place a proof of the Classification. The original proof sought only to make sure the literature actually contains all the pieces of some program purporting to prove the Classification Theorem. The second effort aims to produce a more readable treatment that inspires a higher level of confidence.

There is a two-volume exposition of part of the original proof by Gorenstein in [G1] and [G2]. Gorenstein's books do not attempt to give details of the proof, but only to give an outline of what is entailed. Further, Gorenstein died before completing the third volume of the series.

The largest part of the second effort is the program begun by Gorenstein, Lyons, and Solomon (GLS). While this program does not attempt to address all parts of the proof, if completed as envisioned, it would deal with most parts. The work is being published by the American Mathematical Society (AMS), and at the time I write this article, five volumes in the series have appeared in [GLS]. In [GLS] you will also find references to other parts of the second effort.

I have described the Classification as a theorem, and at this time I believe that to be true. Twenty years ago I would also have described the Classification as a theorem. On the other hand, ten years ago, while I often referred to the Classification as a theorem, I knew formally that that was not the case, since experts had by then become aware that a significant part of the proof had not been completely worked out and written down. More precisely, the so-called "quasithin groups" were not dealt with adequately in the original proof. Steve Smith and I worked for seven years, eventually classifying the quasithin groups and closing this gap in the proof of the Classification Theorem. We completed the write-up of our theorem last year; it will be published (probably in 2004) by the AMS. Later I will state the result; it should be viewed as part of both the original proof and the second effort.

It is time for some specifics: The proof of the Classification proceeds by studying the so-called *local subgroups* of G. Let p be a prime. A *p-local subgroup* of G is the normalizer of a nontrivial p-subgroup of G.

Let G be our minimal counterexample. The proof of the Classification can be thought of as made up of two steps:

Step 1. Prove the local structure of G resembles that of some $\bar{G} \in \mathcal{K}$.

Step 2. Use the resemblance in Step 1 to prove $G \cong \bar{G}$.

Step 2 for the sporadic groups is one of the parts of the second effort that [GLS] does not address. However, there has been quite a bit of progress in improving the treatment of Step 2 since the original proof. Many of the original proofs of the existence and uniqueness of the sporadic groups were machine aided, and the mathematics involved was often unnecessarily complicated. In the last twenty years, methods with the flavor of combinatorial group theory and/or algebraic topology have emerged that provide simpler, more conceptual, treatments of uniqueness in Step 2 and have eliminated almost all computer calculations. I believe, however, that some of the old computer-aided existence and uniqueness proofs have not been superseded; e.g. the proof of the uniqueness of Thompson's group and the existence proof for O'Nan's group are probably still machine aided. I will not say more than that about Step 2 but instead will focus on Step 1.

The generic finite simple group is a finite group of Lie type. Each such group G is described via a representation as a linear group, say $G \leq GL(V)$ for some finite-dimensional vector space V over some finite field F. Thus G has a characteristic which is a prime: The characteristic p of F. Similarly the Lie rank of G is a measure of the "size" of G, and for our purposes we can think of the Lie rank as roughly the dimension of V. Finally, given a prime r, the r-local structure of G is qualitatively different when $r = p$ from when $r \neq p$. To implement Step 1 we must translate these notions for linear groups into related notions defined in abstract groups.

Let p be a prime, G a finite group, and H a p-local subgroup of G. Define H to be of *characteristic p* if

$$C_H(O_p(H)) \leq O_p(H),$$

where $O_p(H)$ is the largest normal p-subgroup of H and for $U \subseteq G$, $C_H(U)$ is the subgroup of all elements of G commuting with each element of U. Define G to be of *characteristic p-type* if each p-local subgroup of G is of characteristic p, and define G to be of *even characteristic* if for each 2-local subgroup H of G containing a Sylow 2-subgroup of G, H is of characteristic 2. It turns out that each group of Lie type and characteristic p is of characteristic p-type.

For reasons I will not go into, the prime 2 plays a special role in the local theory of finite simple groups. (For example, by the Feit-Thompson Theorem [FT], nonabelian finite simple groups are of even order.) Thus in the original proof the following partition appears:

Case I. The minimal counterexample G is of characteristic 2-type.

Case II. G is *not* of characteristic 2-type.

The generic group appearing in Case I is a group of Lie type and characteristic 2, while almost all other simple groups appear in Case II.

There is also a partition according to size that corresponds roughly to the notion of size for groups of Lie type: Given a prime p and a finite group G, define the *p-rank* $m_p(G)$ of G to be the maximal dimension of an abelian subgroup of G of exponent p, regarded as a vector space over the field of order p. In Case II the "size" of G can be taken to be the 2-rank $m_2(G)$ of G. In Case I the size is the parameter $e(G)$ defined by Thompson in the n-group paper [T] (the model for all later work on Case I):

$$e(G) = \max\{m_p(H) : H \text{ is a 2-local of } G \text{ and } p \text{ is an odd prime}\}.$$

Define G to be *quasithin* if $e(G) \leq 2$. The "small" groups in Case I are the quasithin groups. Thus in the original proof we have four blocks in our partition corresponding to the large and small groups of characteristic 2-type and to the large and small groups not of characteristic 2-type. The quasithin groups of characteristic 2-type constitute one of the four blocks.

In the GLS program, one of the partitions is changed by altering the definition of "characteristic". I will not give the GLS definition, since it is technical. However, to accommodate the GLS change, Steve Smith and I also work with a different notion of characteristic in our study of quasithin groups: Recall that G is of even characteristic if $C_H(O_2(H)) \leq O_2(H)$ for each 2-local containing a Sylow 2-subgroup of G. Define a finite group G to be a *QTKE-group* if G is quasithin of even characteristic and each proper simple section of G is in $\mathcal{K}$.

Classification of QTKE Groups. *(Aschbacher-Smith) Let G be a nonabelian simple QTKE-group. Then G is isomorphic to one of the following:*

1. A group of Lie type of characteristic 2 and Lie rank at most 2, but not $U_5(q)$.

2. $L_4(2)$, $L_5(2)$, $Sp_6(2)$, or $U_5(4)$.

3. The alternating group A_9.

4. $L_2(p)$ for p a Mersenne or Fermat prime, $L_3(3)$, $U_3(3)$, $L_4(3)$, $U_4(3)$, or $G_2(3)$.

5. One of 11 sporadic groups: a Mathieu group, a Janko group other than J_1, HS, He, or Ru.

The proof of this theorem will be published by the AMS in two volumes in [AS]. It is roughly 1,200 pages in length. In part this reflects the complexity of the proof, but it also reflects a style of exposition that includes more detail than one usually finds in the original proof of the Classification, and it reflects our decision to keep our treatment as self-contained as possible. Indeed, one of the two volumes is devoted to group-theoretic infrastructure, such as proofs of folk theorems and facts about $\mathcal{K}$-groups.

To my knowledge the main theorem of [AS] closes the last gap in the original proof, so (for the moment) the Classification Theorem can be regarded as a theorem. On the other hand, I hope I have convinced you that it is important to complete the program by carefully writing out a more reliable proof in order to minimize the chance of other gaps being discovered in the future. Thus our discussion of the status of the Classification would not be complete without some indication of what remains to be done in that program.

Recall that the condition that G be of even characteristic is weaker than the condition that G be of characteristic 2-type; thus if one changes the partition in the original proof to a division based on the even characteristic condition, more groups appear in Case I. GLS work with so-called groups of *even type*; again, this condition is weaker than the characteristic 2-type condition, so more groups also appear in Case I in their partition, and hence this part of the problem becomes more difficult. However, as a corollary to our main theorem, Steve Smith and I also determine in [AS] the quasithin groups of even type, the result on quasithin groups required in the GLS approach to the Classification. Thus any difficulties involved in treating the larger class of groups in the extended Case I have at least been overcome for the small groups in Case I.

In the original proof of the Classification, chronologically Case II was treated before Case I, more time was spent dealing with Case II, and more people worked on this case. Probably as a result, the treatment of Case II in the original proof is in better shape than the treatment of Case I. Volume 6 in the [GLS] series treats the small groups in (their redefined) Case II. After a certain point, GLS treat the large groups in Cases I and II together; this work is begun in volume 5 of [GLS] and will be completed in the next few volumes. Thus the part of the second effort that is furthest from completion is the initial phase of Case I.

The approach used to treat large groups in Case I is to focus on p-locals for odd primes p but to keep 2-locals in the picture. This leads us to the following definitions: Given a 2-local H, define

$$\sigma(H) = \{p : p \text{ is an odd prime and } m_p(H) > 2\},$$

and let σ be the union of the sets $\sigma(H)$, as H varies over the 2-locals of G. We work with p-locals for $p \in \sigma$.

In the original proof, two problems in Case I required special treatment:

A. The *uniqueness case*: There exists a 2-local H with $\sigma(H) \neq \varnothing$ and H strongly p-embedded in G for each $p \in \sigma(H)$; that is, $|H \cap H^g|$ is prime to p for each $g \in G - H$.

B. The case $e(G) = 3$.

Initially GLS hoped that new methods developed after 1980 (e.g. the so-called *amalgam method*) could be used to treat both cases (A) and (B) for the larger class of groups of even type, and their program was based on the assumption that specialists in those methods would handle the two cases. However, to date this has not happened, so the treatment of cases (A) and (B) in groups of even type remains as perhaps the greatest obstacle to the completion of the second effort. The work on these cases in groups of characteristic 2-type in the original proof is available as a model, but it is not clear how much more difficult the problems are in groups of even type. Gernot Stroth and Inna Korchagina have done preliminary work on (A) and (B), respectively, under the hypothesis that G is of even type. Case (B) requires special treatment, because different signalizer functors (cf. Volumes 1 and 2 of [GLS]) are used in p-rank 3 and p-rank greater than 3.

Finally, in addition to the original proof of the Classification and the second-generation approach of GLS, there is also a third-generation program involving a number of people, particularly Ulrich Meierfrankefeld, Bernd Stellmacher, and Gernot Stroth, that would treat all groups of characteristic 2-type (and perhaps eventually all groups of even characteristic) using the amalgam method. Steve

Smith and I made use of variations of this method in our work in [AS]. It is possible that this approach will give a better treatment of Case I or at least of Case (B).

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