The Difficulties of Kissing in Three Dimensions

Bill Casselman

As the article by Florian Pfender and Günter Ziegler in this issue explains, the kissing number in four dimensions has apparently been shown by Oleg Musin to be 24, after several years of speculation that this was so. The same problem in three dimensions continues to be of considerable interest, even though it was shown as long ago as 1953, by Schütte and van der Waerden, that in three dimensions no more than 12 spheres could be placed in contact with a central thirteenth, all of the same size.

The history of the problem is obscure. It is commonly said that in a discussion that took place in Cambridge the Scottish astronomer and mathematician David Gregory asserted that 13 spheres could be placed in contact with a central sphere, while Isaac Newton claimed that only 12 were possible. Evidence for exactly what was said in this discussion is murky. The first published reference to it that I know of is in the third volume of Newton's correspondence, edited by H. W. Turnbull, which came out in 1961. There is an entry for May 4, 1694, one of several Latin memoranda written about that day by Gregory, summarizing a conversation with Newton on the distribution of stars of various magnitudes. On the question of 12 *versus* 13, the entry does not support what is commonly said. Two distinct possibilities are not mentioned, and the most plausible reading is that Newton himself thought that 13 spheres surrounding a fourteenth was a possibility! More likely, some would think, is that Gregory didn't understand what Newton had said in an apparently rather rapid discourse. Turnbull refers to a more elaborate entry in a notebook of Gregory kept at Christ Church, Oxford, but at least one person's attempt to locate that entry where Turnbull said it should be was unsuccessful. In any event, Turnbull's paraphrase suggests that there is nothing important there not already mentioned in the published memorandum. Other puzzling features of this story are that the 1953 paper by Schütte and van der Waerden refers to a Newton-Gregory discussion, and in 1956 John Leech referred in more detail to the Christ Church notebook. These both appeared several years before the correspondence of Newton appeared in print. What was the source of their information? To paraphrase a familiar dictum of the Mattel Toy Company, history is hard.

R. Hoppe thought he had solved the problem in 1874. Although the first proof now accepted as valid appeared in 1953, Coxeter in 1963 refers to Hoppe's proof as if it were correct. Schütte and van der Waerden make no reference to nineteenth century work, and the first published analysis of Hoppe's mistake that I know of is that by Hales in his 1994 *Intelligencer* article. There are several historical puzzles here, too, about the track of mathematical ideas.

As far as I know, no really simple proof of the result of Schütte and van der Waerden has been found. The one probably most admired is that presented by Leech in a cryptic two-page note, but although his reasoning has been accepted as correct, there are gaps in his exposition, many involving spherical trigonometry no longer generally familiar (for example, Lexell's circle), and I am not aware that anyone has ever written a complete account. To illustrate that as Tom Hales has written, "The subject is littered with faulty arguments and abandoned methods," I can point out that the first edition of the well known book by Ziegler and Martin Aigner included an expansion of Leech's argument that, although usefully filling in many gaps in Leech's exposition, turned out to be erroneous. Rather than patch it up for the second edition, the authors simply removed this chapter, feeling presumably that they didn't want to include so much spherical trigonometry in what they hoped would be a "perfect proof." It would be valuable if someone were to publish an account of Leech's proof that

Bill Casselman is professor of mathematics at the University of British Columbia. He is also graphics editor and a member of the editorial board for the Notices. *His email address is* cass@math.ubc.ca.

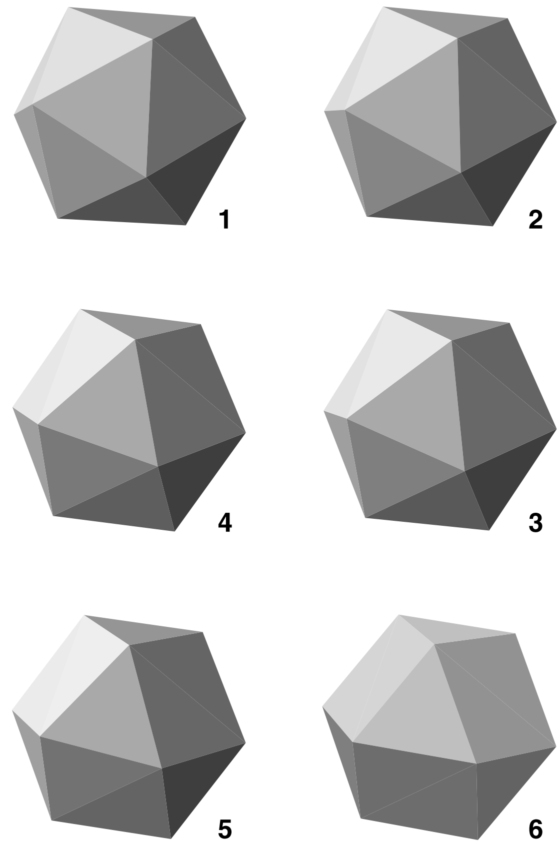
made it accessible to an elementary undergraduate course.

In connection with other problems involving the distribution of spheres in three and four dimensions, new proofs that the kissing number in 3D is 12 have been proposed in recent publications by Wu-Yi Hsiang, Károly Böröczky, Kurt Anstreicher, and finally Oleg Musin. The last of these is particularly interesting, since the 3D case offers an instructive warm-up exercise for the more difficult one in 4D.

If the three-dimensional problem continues to be of interest, one reason is presumably that here is a part of the mathematical universe where the borderline between essential rigour and mere detail in proofs is particularly difficult to perceive. There is a great deal of room in the geometry of three-dimensional configurations in which to dispute over what's necessary and what's tedious.

There are real mathematical difficulties present, however, in addition to psychological ones. In contrast to other dimensions where the exact kissing number is known, in dimension three optimal solutions form a continuum. Interesting things take place in this space of all acceptable configurations. In one configuration the centres of the exterior spheres are positioned loosely at the vertices of a regular icosahedron, and in another they are positioned at the vertices of a regular cuboctahedron. Coxeter mentioned in §3.7 of *Regular Polytopes* that the vertices of an icosahedron can be obtained by dividing the edges of an octahedron according to the golden section, and describes a continuous family of shapes running from the octahedron through a regular icosahedron to a cuboctahedron.

In SPLAG, Conway and Sloane point out that this leads to a path among kissing configurations, and go on in a remarkable discussion to relate this construction to properties of the Mathieu group. There are a number of interesting open questions implicit here, as John Baez has mentioned in one of his web notes—the space of all allowable configurations is at once intriguing and not well understood. There are several indisputable proofs that the kissing number is 12, but it would be very pleasant to see this more clearly than we do now.



Coxeter's deformation of the icosahedron into the cuboctahedron. The cover of this issue shows the corresponding deformation of kissing circles on the sphere.

Further reading

M. Aigner and G. M. Ziegler, *Proofs from THE BOOK*, Springer, 1998 (first ed.) and 2002 (second ed.).

K. Anstreicher, The thirteen spheres: A new proof, *Discrete & Computational Geometry* **31** (2004), 613–625.

J. Baez, This week's find in mathematical physics (Week 20), from a link at `http://math.ucr.edu/home/baez/twf.html`.

K. Böröczky, The Newton-Gregory problem revisited, *Proc. Discrete Geometry*, Marcel Dekker, 2003, pp. 103–110.

J. Conway and N. J. Sloane, *Sphere Packings, Lattices, and Groups*, Springer, 1993 (second ed.) and 1999 (third ed.).

H. S. M. Coxeter, An upper bound for the number of equal nonoverlapping spheres that can touch another of the same size, *Proc. Symp. Pure Math.* VII, 1963, pp. 53–72.

T. Hales, The status of the Kepler conjecture, *Mathematical Intelligencer* **16** (1994), 47–58.

W.-Y. Hsiang, *Least Action Principle of Crystal Formation of Dense Packing Type and Kepler's Conjecture*, World Scientific, 2001.

I. Newton, ***The** Correspondence of Isaac Newton*, H. W. Turnbull (ed.), Oxford, 1961.

G. G. Szpiro, *Kepler's Conjecture*, Wiley, 2002.

I. Todhunter, *Spherical Trigonometry*, Macmillan, 1886.

See also the reference list in the article by Pfender and Ziegler. I would like to thank Tom Hales, George Szpiro, and Günter Ziegler for their help. *—B.C.*