AMS Short Course

The Radon Transform and Applications to Inverse Problems

Organizers: Gestur Olafsson, Louisiana State University, and Todd Quinto, Tufts University

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It is planned that lecture notes will be available to those who register for this course. Advance registration fees are $85 for AMS/MAA members, $108 for nonmembers, and $37 for students/unemployed/emeritus; on-site registration fees are $115 for AMS/MAA members, $140 for nonmembers, and $55 for students/unemployed/emeritus. Registration and housing information can be found in this issue of the *Notices*; see the section "Registering in Advance and Hotel Accommodations" in the announcement for the meetings in Atlanta. The registration form is at the back of this issue.

General Introduction

The mathematical model of X-ray tomography is the Radon transform, named after the Austrian mathematician Johann Radon, who proved important properties of the transform in a seminal 1917 article. Mathematicians and scientists did not know that this pure mathematical transform had applications to medicine and science until after Nobel Laureate Allan Cormack made the first CT (computerized tomography) scanner and he and others developed the mathematics to image objects from this indirect X-ray data. Since then, tomography has become important in pure and applied mathematics, as well as in several branches of applied sciences, in particular diagnostic radiology, nondestructive evaluation, and other forms of image reconstruction. The aim of this short course is to introduce nonspecialists to the basic mathematics and ideas behind tomography. Then several important applications will be described in diagnostic radiology, electron microscopy, radar, and sonar. All talks will be aimed at a general audience.

We will begin with elementary facts about the Radon transform and then introduce important current research areas, including local tomography, 3-D tomography and regularization, interferometric imaging, sampling theory, wavelet methods, and emission tomography. Several special sessions at the AMS meeting will continue the themes introduced in the short course.

The titles of the talks, the names of the speakers, and the abstracts are as follows:

An Introduction to Tomography and Radon Transforms

Todd Quinto, Tufts University

In tomography, one finds information about the inside of objects using indirect data. X-ray tomography has revolutionized diagnostic radiology, and CT scans (in which the body is X-rayed from multiple positions around the body) provide the intricate structure of the inside of the body without exploratory surgery. The classical Radon transform is the mathematical model of X-ray tomography, and it integrates functions over lines, the lines along which the X-rays travel. From these data, inversion formulas give accurate and detailed pictures of the inside of the body.

This will be the first talk of the session, and we will introduce some of the basic mathematics of tomography. We will start by defining the Radon transform and introducing a few fundamental theorems, including the projection slice theorem.

We will then describe some limited data tomographic problems that occur in science. Finally, we plan to introduce microlocal analysis and use it to understand what singularities (object boundaries, etc.) are well imaged from limited data.

Reading List

[1] S. Helgason, *The Radon Transform*, 2nd edition, Birkhauser, Boston, 1999.

[2] F. Natterer and F. Wuebbeling, *Mathematical Methods in Image Reconstruction*, SIAM, New York, Monographs on Mathematical Modeling and Computation, 2001.

[3] E.T. QUINTO, Singularities of the X-ray transform and limited data tomography in R^2 and R^3, *SIAM J. Math. Anal.*, **24** (1993), 1215–1225.

Development of Algorithms in CT

Alfred Louis, Angewandte Mathematik, Universitaet des Saarlandes

The success of CT relies dramatically on the ability to quickly reconstruct the searched-for information from the measured data. Compared with the first commercial scanners, the speed-up in computing time is much more influenced by the new mathematical algorithms than by the new computer hardware.

Hence in this talk we study the principles of the development of algorithms in CT. We start with the simplest case, the 2-D parallel beam geometry leading to the so-called filtered backprojection algorithm. We consider this as a special case of regularization method for inverse problems, the approximate inverse that allows for precomputing independently of the data reconstruction kernels and using invariances of the problem. These tools are then applied to the inversion problems of 3-D cone beam tomography. We present reconstructions from real data and many open problems.

Reading List

[1] A. KATSEVICH, A general scheme for constructing inversion algorithms for cone beam CT, *International Journal on Mathematics and Mathematical Sciences,* **21** (2003), 1305–1321.
[2] A.K. LOUIS, The approximate inverse for linear and some nonlinear problems, *Inverse Problems,* **12** (1996),175–190.
[3] A.K. LOUIS, Filter design in three-dimensional cone beam tomography: circular scanning geometry, *Inverse Problems*, **19** (2003),S31-S40.
[4] F. NATTERER and, F. WUEBBELING, *Mathematical Methods in Image Reconstruction*, vol. 5, SIAM, Philadelphia, PA, 2001.

Generalized Transforms of Radon Type and their Applications

Peter Kuchment, Texas A&M University

Radon-type transforms, in the simplest cases, integrate a function on a Euclidean space over all affine planes of a given dimension. In many types of medical, geophysical, and industrial tomography problems, the measured data are a Radon transform of an unknown distribution of certain quantity of interest (e.g., CAT scan data produces integrals of the density of the body over "all" lines in a plane). Hence invertibility of the transform, availability of inversion formulas and/or algorithms, and stability of the inversion (that is, not being very sensitive to errors in the data) are crucial for many applications.

In the last couple of decades, many new examples have been found in which one needs to invert some generalized transforms of Radon type. This means that the transform involves certain weighted integrals and/or the manifolds over which the integration is conducted are not flat (e.g., in some applications integration over spheres is involved). The lecture will contain some examples of such transforms arising in applications, as well as discussion of the available results about them.

References

[1] F. NATTERER, *The Mathematics of Computerized Tomography*, Wiley, New York, 1986.
[2] F. NATTERER and F. WUEBBELING, *Mathematical Methods in Image Reconstruction*, Monographs on Mathematical Modeling and Computation, vol. 5, SIAM, Philadelphia, PA, 2001.

Tomography and Sampling Theory

Adel Faridani, Department of Mathematics, Oregon State University

Computed tomography entails the reconstruction of a function from measurements of its line integrals. In this talk we explore the question: How many and which line integrals should be measured in order to achieve a desired resolution in the reconstructed image? Answering this question may help to reduce the amount of measurements and thereby the radiation dose, or to obtain a better image from the data one already has. Our exploration leads us to a mathematically and practically fruitful interaction of Shannon sampling theory and tomography. For example, sampling theory helps to identify efficient data acquisition schemes, provides a qualitative understanding of certain artifacts in tomographic images, and facilitates the error analysis of some reconstruction algorithms. On the other hand, applications in tomography have stimulated new research in sampling theory, e.g., on nonuniform sampling theorems and estimates for the aliasing error.

References

[1] A. FARIDANI, Sampling theory and parallel-beam tomography, *Sampling, Wavelets, and Tomography*, (J.J. Benedetto and A. I. Zayed, eds.), Birkhauser, Boston, 2004, pp. 225–254.
[2] A. FARIDANI, Introduction to the mathematics of computed tomography, *Inside Out: Inverse Problems and Applications*, G. Uhlmann (ed.), MSRI Publications, Vol. 47, Cambridge University Press, 2003, pp. 1–46.
[3] www.oregonstate.edu/~faridana/preprints/preprints.html.

Inverse Problems in Pipeline Inspection

Peter Massopust, Tuboscope Pipeline Services, Houston, TX

One of the techniques to detect defects and anomalies in pipelines is based on the magnetization of the pipe wall. Anomalies and defects in the pipe cause changes in the induced magnetic field, and these changes are measured by a set of sensors. From these measurements, the shape and type of defect needs to be inferred. Several factors (e.g., sensor sensitivity, probability of detection, noise) can impede these measurements, making the solution of the inverse problem *indirect measurements_geometry* more difficult. In this introductory lecture, we discuss the fundamental model to solve the inverse problem and show how B-spline and wavelet techniques can be successfully applied to remedy measurement impediments.

References

[1] J. D. JACKSON, *Classical Electrodynamics*, John Wiley & Sons, 3rd edition, 1998.
[2] I. DAUBECHIES, *Ten Lectures on Wavelets*, SIAM, 1992.
[3] C. DE BOOR, *A Practical Guide to Splines*, Springer-Verlag, 2001.

Coherent Interferometric Array Imaging in Random Media

Liliana Borcea, Computational & Applied Mathematics, Rice University

In important applications, such as ultrasound medical imaging, foliage or ground penetrating radar, land and shallow water mine detecting, etc., one seeks to detect and image small or extended scatterers (reflectors) embedded in inhomogeneous, cluttered media. Such media can be modeled as randomly inhomogeneous, with properties such as acoustic impedance having a deterministic large-scale variation, assumed known, and an additional weak, small-scale variation that is unknown and is represented by a random function of space. The strong scatterers embedded in such media are to be imaged with an array of transducers that emit acoustic pulses and record the backscattered echoes.

Traditional array imaging methods known as synthetic aperture sonar, Kirchhoff Migration, etc., are well understood and work well in known media. However, these methods fail in random media and new ideas must be explored in order to obtain reliable images. I will describe a new, coherent interferometric approach to imaging in clutter, which is based on an asymptotic stochastic analysis of wave propagation in random media, in regimes with strong multipath. To achieve stable results, this method uses cross-correlations of nearby traces recorded at the array, the interferograms. We also exploit the existence of a frequency coherence band in order to achieve good resolution of the images. Naturally, the spatial and frequency coherence of the data at the array depend on the random medium, and we show how they quantify explicitly the resolution of the images. The efficiency and robustness of the proposed method in clutter will be illustrated with several numerical results.

References

[1] P. BLOMGREN, G. PAPANICOLAOU, and H. ZHAO, Super-resolution in time-reversal acoustics, *Journal of the Acoustical Society of America*, **111** (2002), 238–248.
[2] L. BORCEA, G. PAPANICOLAOU, and C. TSOGKA, Theory and applications of time reversal and interferometric imaging, *Inverse Problems*, **19** (2003), S134–164.
[3] L. BORCEA, G. PAPANICOLAOU, C. TSOGKA, and J. BERRYMAN, Imaging and time reversal in random media, *Inverse Problems*, **18**, No. 5, (2002), 1247–1279.
[4] G. PAPANICOLAOU, L. RYZHIK, and K. SOLNA, Statistical stability in time reversal, *SIAM J. Applied Mathematics*, in press. Available from G. Papanicolaou's webpage: http://georgep.stanford.edu/~papanico/pubs.html.