Double Cusp Group

David J. Wright

People have long been fascinated with repeated patterns that display a rich collection of symmetries. The discovery of hyperbolic geometries in the nineteenth century revealed a far greater wealth of patterns, some popularized by Dutch artist M. C. Escher in his *Circle Limit* series of works. The cover illustration on this issue of the *Notices* portrays a pattern which is symmetric under a group generated by two Möbius transformations $a(z)$ and $b(z)$ of the form $\frac{\alpha z+\beta}{\gamma z+\delta}$ where α, β, γ, and δ and z are all complex numbers. These are not distance-preserving, but they do preserve angles between curves and they map circles to circles.

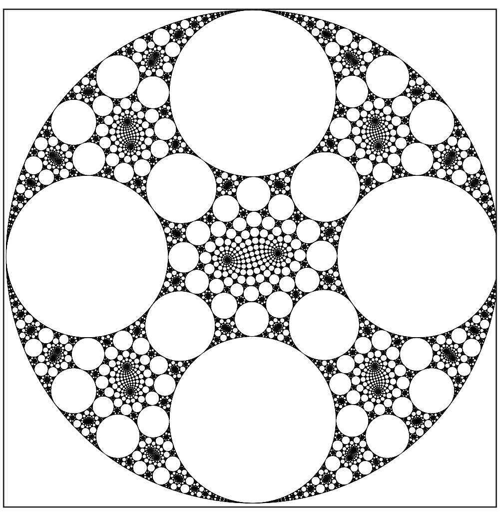

Figure 1.

This black lacework (Figure 1) is the set of 'limit points' of the group, meaning any point that may be approached by an infinite sequence of distinct transformations applied to some other point. Euclidean wallpaper patterns have only one limit point at infinity. Escher's *Circle Limit* works have a circle of limit points. The limit set here is composed of infinitely many circles and the accumulation points of these circles.

Figure 2 is a finite web of tangent disks from the cover picture that defines how the transformations *a* and *b* operate. The violet disks are moved from one to the next by the transformation *a*, indicated by red arrows. The two light blue disks are invariant by the transformation *b*, shown by the blue arrows. Any Möbius transformation that leaves invariant two tangent disks is called 'parabolic', and has a unique fixed point at the point of

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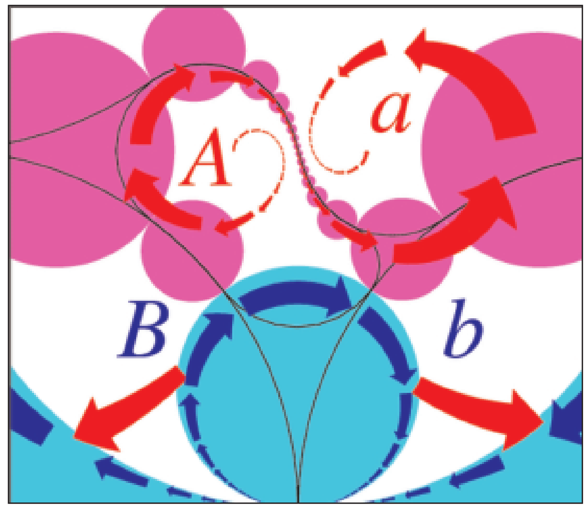

Figure 2.

tangency. The trace of the matrix corresponding to b must be ± 2, giving a polynomial condition on the generators a and b. On the other hand, the a transformation moves points in infinite double spirals from its 'repelling' fixed point to its 'attractive' fixed point.

The complement of the circle web consists of four white regions or 'blobs' labelled a, A, b and B. We stipulate that the a transformation map the

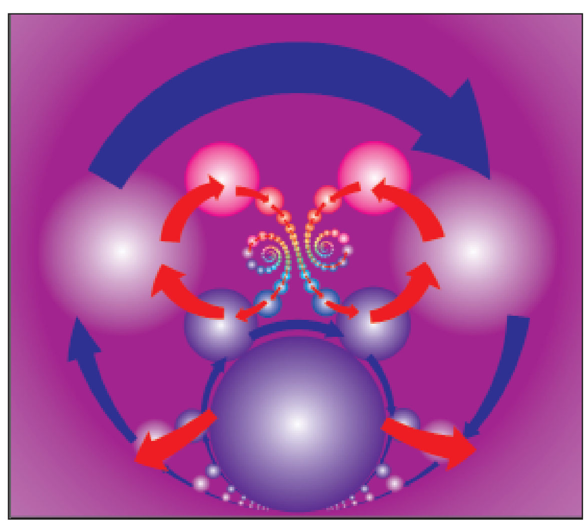

Figure 3.

ring of circles around the A blob into the ring of circles around the a blob in reverse order, and similarly for the b transformation and blobs. Also, disks must be mapped to disks of the same color.

Figure 3 shows some closed loops of arrows between the disks. Following the arrows corresponding to the composition $a \circ b^{-1} \circ a^{-1} \circ b$, namely, a forward blue arrow, then a backward red arrow, then a backward blue arrow, then a forward red arrow, we see that this transformation fixes the large outer 'disk' containing ∞ (which we'll denote D_1) as well as the largest purple disk at the left (which we'll denote D_2). That implies that $a \circ b^{-1} \circ a^{-1} \circ b$ is also parabolic, another polynomial condition on a and b.

Furthermore, if we start at D_2 and proceed by exactly 15 applications of a (red arrows), we arrive at the symmetrical large purple disk at the right, and then we can jump back to D_2 by b^{-1} (backwards blue arrow). This same composition also fixes the disk $a^{-1}(D_2)$ just one red arrow prior to D_2, implying that $b^{-1} \circ a^{15}$ is parabolic. These parabolic conditions uniquely determine the pattern (up to conjugacy).

The coloring of disks works as follows. Applying b doesn't change the color; applying a changes the color of the disk to the next one in a cycle of fifteen colors. That this is consistent depends on some properties of this group.

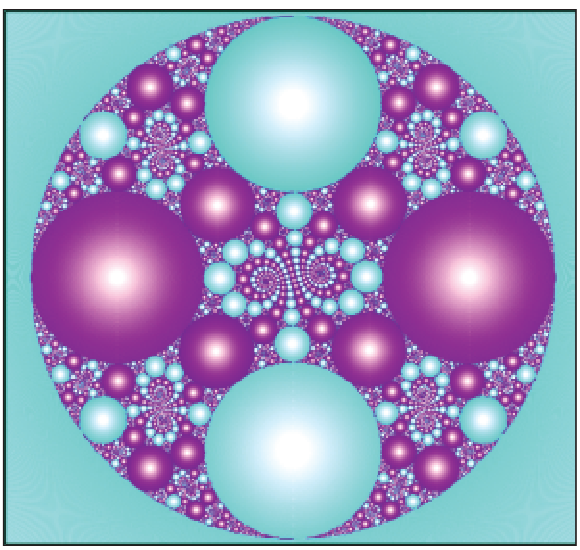

Figure 4.

Figure 4 simply shows the two equivalence classes of disks under the action of the group. It extends the coloring of the disks in the original circle web. Each disk is fixed by a sizeable subgroup of transformations. For example, we have seen the outer disk D_1 containing ∞ is invariant under b and $a \circ b^{-1} \circ a^{-1} \circ b$; these two 'words' generate the stabilizer subgroup of this disk. All the words in this subgroup have the sum of the exponents of the a terms equal to 0. Thus, the color of the image of this disk under any word in a and b is determined just by the sum of the exponents of the a terms.

Similarly, the disk D_2 has stabilizer generated by $a \circ b^{-1} \circ a^{-1} \circ b$ and $b^{-1} \circ a^{15}$. Any word in this subgroup has the exponents of the a's summing to a multiple of 15. Hence, our coloring will be consistent if we use 15 colors.

Choosing the same color for the D_1 and D_2, we see the following rule throughout the pattern: two tangent disks belong to different classes if and only if they have the same color.

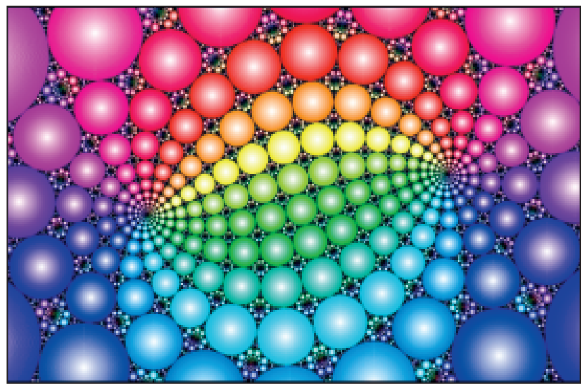

Figure 5.

Figure 5 is a zoom into the center of the picture, where the fixed points of the a transformation appear like hypnotic eyes. The title "double cusp group" refers to this group's origin as an extreme 'deformation' of two-generator 'quasifuchsian' groups. Some discussion of this may be found in Chapter 9 of [2].

About similar kinds of groups and their limit sets, Klein wrote in 1894 [1]:

"The question is, what will be the configuration formed by the totality of all the circles, and in particular what will be the position of the limiting points. There is no difficulty in answering these questions by purely logical reasoning; but the imagination seems to fail utterly when we try to form a mental image of the result."

All the pictures were rendered using a program "kleinian" written by the author in collaboration with David Mumford. This note was prepared while the author was on sabbatical at the University of Oklahoma in Norman, and the author wishes to thank the department there for indulging his activities.

References

[1] Felix Klein, Lectures on Mathematics, American Math. Soc., 2000.

[2] David Mumford, Caroline Series, and David Wright, Indra's Pearls: The Vision of Felix Klein, Cambridge University Press, 2002.