Book Review

Kepler's Conjecture and Hales's Proof

A Book Review by Frank Morgan

Kepler's Conjecture: How some of the greatest minds in history helped solve one of the oldest math problems in the world
George G. Szpiro
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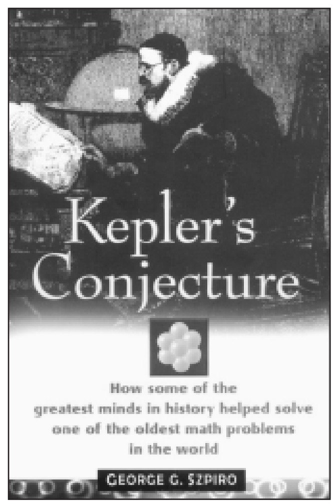
In 1611 Kepler conjectured that the standard way of packing unit spheres is actually the densest. In 1993 the *International Journal of Mathematics* published a purported proof by Wu-Yi Hsiang. The incredible features were that the proof considered only close neighbors (centers within 2.18) and used mainly trigonometry. As far as I know, there are no counterexamples to the method, and the first observed mistakes have been repaired; but Hsiang's proof has not been generally accepted by the mathematics community. Thomas Hales, soon to announce his own proof, published a sharp criticism in the *Mathematical Intelligencer* (1994), soon followed by a rejoinder by Hsiang (1995). The review by Gábor Fejes Tóth in *Mathematical Reviews* provides interesting reading.

Hales submitted his proof to *Annals of Mathematics*. In his expository article "Cannonballs and Honeycombs" in the April 2000 *Notices*, he reported that

> a jury of twelve referees has been deliberating on the proof since September 1998.

They did have a tough job. It was a momentous result. After the controversies over Hsiang's published proof, they had to be careful. And it was a



tough body of work to referee, consisting of six papers, including the thesis of Hales's Ph.D. student Samuel Ferguson. The proof involved extensive computer analysis. Many cases initially would have required more computation time than the age of the universe. The proof had to be continually modified and aided by intricate analysis and geometry. In their paper on "A Formulation of the Kepler Conjecture", Ferguson and Hales wrote:

> As our investigations progressed, we found that it was necessary to make some adjustments. However, we had no desire to start over, abandoning the results of "Sphere Packings I" and "Sphere Packings II." "A Formulation" gives a new decomposition of space [but] shows that all of the main theorems from "Sphere Packings I" and "Sphere Packings II" can be easily recovered in this new context with a few simple lemmas.

After years of effort, the referees gave up. Meanwhile, Hales and the world were waiting for the referees' conclusions. *Annals* finally decided on an unprecedented course of action: to publish the work with a disclaimer that the referees had been unable to verify the proof.

Frank Morgan is Webster Atwell Class of 1921 Professor of Mathematics at Williams College. His email address is Frank.Morgan@williams.edu.

At the Joint Mathematics Meetings in Baltimore in January 2003, Hales received the Chauvenet Prize of the Mathematical Association of America for his *Notices* article. In his acceptance speech, he read from a letter he received from the editors of the *Annals*, leaving the impression that they would be unable to publish his result. According to Hales, *Annals* had written:

> The news from the referees is bad…. They have not been able to certify the correctness of the proof, and will not be able to certify it in the future, because they have run out of energy…. One can speculate whether their process would have converged to a definitive answer had they had a more clear manuscript from the beginning, but this does not matter now. [S]

In the lively discussions after the prize ceremony, it was apparent that Hales was not at all satisfied with the *Annals'* delays and with its eventual decision to publish the work with a disclaimer.

The *Annals* then decided to call on another referee, who got Hales to reorganize his papers into more readable, checkable mathematics. *Annals* now intends to publish without disclaimer a single paper with the overall mathematical strategy of the proof. The entire mathematical proof, in six papers, is to appear in a special issue of *Discrete and Computational Geometry*, edited by some of the referees. Some of the computer programs and data will be on an *Annals* website. Some further revising and refereeing may occur before the papers are in final form and accepted, although I think it unlikely that even all of the mathematics can be checked by then, never mind the computer programs.

Meanwhile, Hales has launched a worldwide cooperative project called "Flyspeck" (based on the letters FPK, for "Formal Proof of Kepler") to produce a verification of the proof by computer, which sounds to me orders of magnitude harder than a check by referees.

The Hexagonal Honeycomb

Shortly after Hales finished his proof of Kepler's Conjecture, Denis Weaire recommended to him an even older problem, the Hexagonal Honeycomb Conjecture. It says that regular hexagons, as in Figure 1, provide the least-perimeter way to partition the plane into unit areas. Widely believed and often asserted as fact, even by such notables as Hermann Weyl [W], it was the longest standing open problem in mathematics, going back thousands of years. Around 36 BC, before his death, Marcus Terentius Varro wrote an epistle "On Agriculture" [V] to his young wife on how to take care of their farming estate, including honeybees. He gave two

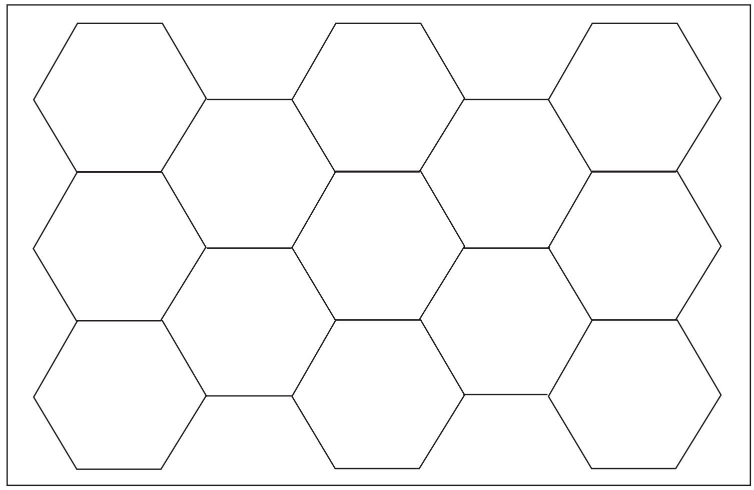


Figure 1. Hales's Hexagonal Honeycomb Conjecture says that regular hexagons provide the least-perimeter way to partition the plane into unit areas.

easons for the hexagonal shape in their honeycombs: first, that a bee has six feet; second,

> The geometricians prove that this hexagon inscribed in a circular figure encloses the greatest amount of space.

Actually the Greek mathematician Zenodorus (200 BC) probably had considered only hexagons, triangles, and parallelograms [He]. Varro's knowledge of bees was not perfect. He also observed that

> They follow their king wherever he goes.

The fact that the leader is not a king but a queen was not discovered until the seventeenth century.

So Weaire recommended the Hexagonal Honeycomb problem to Hales: "Given its celebrated history, it seems worth a try." Hales promptly dispatched it in under a year. "In contrast with the years of forced labor that gave the proof of the Kepler Conjecture, I felt as if I had won a lottery." (Quotations from the book under review, Chapter 14).

One major difficulty in proving regular hexagons optimal is that the result is not true locally. A hexagon is not the least-perimeter way to enclose unit area; a circle is. Of course you cannot partition the plane into unit circular regions. Figure 2 shows the best-known ways to enclose and separate $3 \le n \le 8$ unit areas. For the case $n = 7$, there is a hexagon at the center, and larger such clusters have approximations of regular hexagons near the center; but for even the largest computed clusters, a true regular hexagon appears only in an occasional, especially symmetric case, and then only as the single region at the center.

For a single region, a circle is best, but its favorable outward convexity would cause adjacent regions to have unfavorable inward concavity, so such outward convexity should carry a penalty, and inward concavity a corresponding credit.

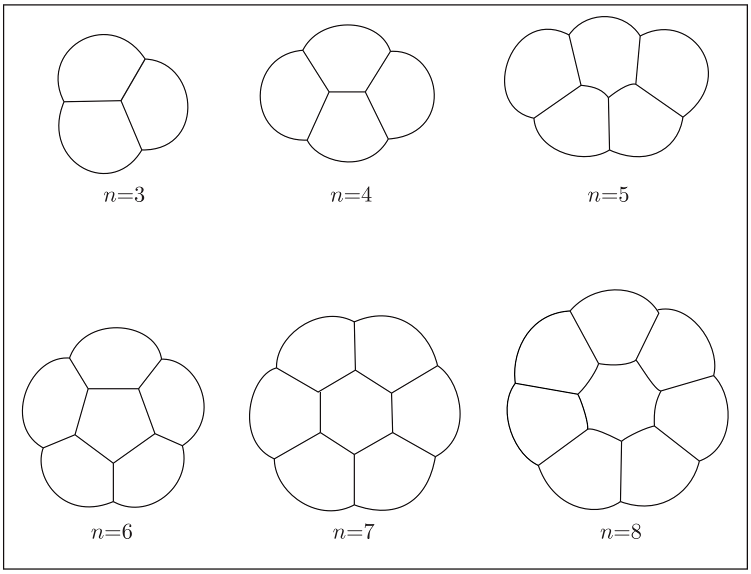


Figure 2. Optimal finite clusters as computed by Cox et al. [C] .

Similarly, polygons with more than six edges can do better than hexagons, but by Euler the average number of edges should be six, so that extra edges should also carry a penalty and fewer edges a credit. Using such penalties and credits, Hales created a new problem in which the regular hexagons are best locally as well as globally. Because globally the penalties and credits must all cancel out, hexagons also solve the original Hexagonal Honeycomb Conjecture.

Here is Hales's local theorem, penalties and all:

Hexagonal Isoperimetric Inequality *[H, Thm. 4]. Let P_0 denote the perimeter of a regular hexagon of unit area. Consider another curvilinear planar polygon, of N edges, unit area, and perimeter P. For each edge, let a_i denote how much more area is enclosed than by a straight line with the same endpoints; truncate a_i so that $-1/2 \leq a_i \leq 1/2$. Then*

$$P/P_0 \geq 1 - .5\Sigma a_i - c(N-6), \tag{1}$$

with for example $c = .0505/2\sqrt[4]{12} \sim .013$, with equality only for the regular hexagon.

Given inequality (1), the idea of the proof of the Hexagonal Honeycomb Conjecture is to sum over all hexagons and let the penalty terms cancel out.

There are of course some technical difficulties. There are infinitely many regions. A region need not be connected. Even the existence of a best partition of the plane was open.

The proof of the inequality (1) involves careful consideration of maybe a dozen cases and subcases, depending for example on whether truncation actually occurred in the definition of a_i and on the size of the penalty terms.

Hales's Proof of Kepler's Conjecture

Just as for the Hexagonal Honeycomb Conjecture, a major difficulty in proving Kepler's Conjecture is that the result is not true locally. The densest way to pack spheres around one central sphere is modeled on the regular dodecahedron, but such an arrangement cannot be continued, because dodecahedra do not tile space.

Hales added local penalties and credits (which cancel out globally) to produce a new problem for which the standard packing would even locally beat the dodecahedral and all other packings. The appropriate penalties are very hard to find. Simple convexity and extra faces do not work; for starters, there is no formula for the average number of faces of a polyhedral partition of space. Earlier workers had tried to fix on features of the associated polyhedral partition into so-called Voronoi cells. (Each Voronoi cell consists of the set of points in space closer to the center of one particular sphere than to the center of any other.) Hales originally had the idea of using instead features of the so-called Delaunay triangulation, with vertices at the centers of the spheres, dual to the Voronoi decomposition. His main conceptual breakthrough may have been when he decided to use both.

Unlike for the planar Hexagonal Honeycomb Conjecture, there were thousands of cases to check, some too difficult for the computer. As he advanced to more and more difficult cases, Hales had to make intricate revisions of the penalties to get the proof to work. (This makes the proof all the harder to check.) The set of penalties finally used was arrived at in collaboration with Ferguson, whose Ph.D. thesis handled the most difficult case.

So despite similarities, Kepler's Conjecture on sphere packing in $\mathbf{R}^3$ was orders of magnitude more difficult than the Hexagonal Honeycomb Conjecture in $\mathbf{R}^2$, mainly because $\mathbf{R}^3$ provides so many more geometric possibilities than $\mathbf{R}^2$. But it was easier in one aspect: it is a *packing* problem, whereas the Honeycomb Conjecture is a *partitioning* problem. The optimal two-dimensional *packing*, with six circles fitting perfectly around every circle, is relatively easy and was proved in 1890 by Thue. (See Klarreich [K] for a beautiful account.) Packing problems are in general much easier because you "just" have to determine where to put the centers of the circles or spheres. For partitioning problems, you have to find the shape or shapes of the regions: hexagons in $\mathbf{R}^2$, still open in $\mathbf{R}^3$. Indeed, in $\mathbf{R}^3$, not only the shape and arrangement of the regions but even the existence of a perimeter-minimizing partition are open. In 1887 Lord Kelvin conjectured that certain relaxed truncated octahedra provide the best partition of $\mathbf{R}^3$ into unit volumes. Kelvin's conjecture was *disproved* in 1994 by D. Weaire and R. Phelan, who exhibited a better partitioning into two different shapes: regular dodecahedra and tetrakaidecahedra with 12 pentagonal faces and 2 hexagonal faces. Proving the Weaire-Phelan structure optimal looks perhaps a century beyond current mathematics to me, but I understand that Hales is already thinking about it.

Szpiro's Book

Szpiro's book *Kepler's Conjecture* is a rich account of both the mathematics and the people involved. The story begins:

> Somewhere toward the end of the 1590s, stocking his ships for yet another expedition, [Sir Walter] Raleigh asked his sidekick and mathematical assistant Thomas Harriot to develop a formula [for the number] of cannonballs in a given stack….

This inspired Harriot's letter to Kepler, and Kepler's Conjecture of 1611 on the densest way to pack spheres in space. In 1831 Gauss proved the result for regular (lattice) packings in a book review. In 1900, Hilbert included the problem in his famous list for the coming century. Every character who appears in the book gets a short biography. In general these biographies are quite interesting. I must admit, however, that in some exciting sections I would get a bit scared that still another character might appear and delay the story with yet another little biography. The appropriately extensive section on Kepler was good, but I thought that the story of his predecessor Tycho Brahe was more than necessary. An account of Andrew Odlyzko's life, including a joint paper with Neil Sloane, leads not only to a biography of Sloane but also to a biography of *his* eccentric and brilliant coauthor John H. Conway, and then to Conway's work in group theory and the Monster Group, and then to Conway's other work, including the book *On Being a Department Head*, actually by the other, John *B.* Conway. (Wouldn't it be interesting, though, to see someone as brilliant and allegedly disorganized as John H. Conway as Princeton department head?)

Incredibly, Szpiro has a lot more related biographical material at `http://www.GeorgeSzpiro.com` (where it belongs, probably along with the mathematical appendices to the book).

Although Szpiro is a published Ph.D. in mathematical economics, the book has all the virtues one would find in a book by a scholarly nonmathematician. Lots of detailed mathematics is covered in a descriptive, interesting, and understandable way. The book includes the famous controversy between Gregory and Newton on whether 13 or merely 12 spheres can touch ("kiss") a fixed central sphere, finally settled in Newton's favor (it's 12) by Schütte and van der Waerden in 1953. Szpiro adds:

> By the way, in 128-dimensional space there exists a grid that allows 218 billion balls to kiss one ball in the center. Quite a crowd, you may say. But if one doesn't care much about neatness, there is a nonlattice arrangement that allows at least 8,863,556,495,104 balls…. Neither of these numbers is thought to be the last word on the subject, however.

Szpiro makes a few amusing gaffes. He reports that according to Gödel's Incompleteness Theorem, "Both a statement and its opposite may be true simultaneously." He calls nonconvex tiles "concave". He misrepresents duality as the trivial observation that minimizing loss is the same as maximizing gain. He calls a certain algorithm "the simplex" instead of "the simplex method". He says that with "Lindemann's proof in 1882 that π is a trancendental number, it was established that this number has infinitely many digits," whereas that conclusion actually follows from mere irrationality. He mistakenly calls the face-centered-cubic (FCC) and hexagonal-close-pack (HCP) sphere packings identical (whereas a correct picture of HCP on page 23(b) would have the three spheres on top rotated by 60 degrees). Both packings arise from stacking identical layers modeled on the hexagonal honeycomb. There are two choices for placing each new layer, yielding even among periodic packings infinitely many essentially equivalent solutions to the sphere packing problem. If you find this topic interesting, you have Szpiro to thank, not only for including it in his book, but also for generously sending me, in response to a draft of this review, this "one more gaffe".

On the whole, *Kepler's Conjecture* is a wonderful book, chock full of interesting mathematics, biography, and drama. I enjoyed and learned a lot from every section, as would, I think, anyone interested in mathematics.

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