Quiver Representations

Harm Derksen and Jerzy Weyman

Introduction

A *quiver* is just a directed graph.[1] Formally, a quiver is a pair $Q = (Q_0, Q_1)$ where Q_0 is a finite set of vertices and Q_1 is a finite set of arrows between them. If $a \in Q_1$ is an arrow, then ta and ha denote its *tail* and its *head,* respectively.

Let us fix a quiver Q and a base field K. Attach a finite dimensional vector space to each vertex of Q and a linear map to each arrow (with the appropiate domain and codomain). Then this is called a *representation of* Q. Formally, a representation V of Q is a collection

$$\{V_x \mid x \in Q_0\}$$

of finite-dimensional K-vector spaces together with a collection

$$\{V_a : V_{ta} \to V_{ha} \mid a \in Q_1\}$$

of K-linear maps. If V is a representation of Q, then its *dimension vector* d_V is the function $Q_0 \to \mathbb{N}$ defined by $d_V(x) = \dim_K(V_x)$ for all $x \in Q_0$. Here $\mathbb{N} = \{0, 1, 2, \dots\}$ denotes the set of nonnegative integers. The set of all possible dimension vectors is $\mathbb{N}^{Q_0}$. Here are a few typical examples of quiver representations.

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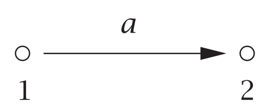
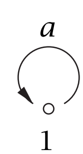
[1] *The underlying motivations of quiver theory are quite different from those in the traditional graph theory. To emphasize this distinction, it is common in our context to use the word "quivers" instead of "graphs".*

Example 1. *A representation of the quiver*



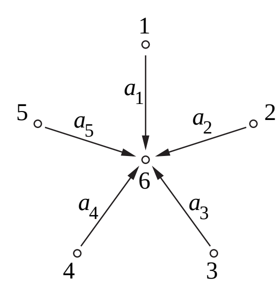
is a collection of two finite-dimensional vector spaces V_1, V_2 *together with a linear map* $V_a : V_1 \to V_2$.

Example 2. *A representation of the loop quiver*



is a vector space V_1 *together with an endomorphism* $V_a : V_1 \to V_1$.

Example 3. *A representation of the star quiver*



is a collection of six vector spaces $V_1, V_2, \dots, V_6$ *together with five linear maps* $V_{a_i} : V_i \to V_6$, $i = 1, 2, \dots, 5$. *If all maps are injective, then we can view such a representation as a subspace configuration.*

If V and W are two representations of Q, then a *morphism* $\phi : V \to W$ is a collection of K-linear maps

$$\{\phi_x : V_x \to W_x \mid x \in Q_0\}$$

such that the diagram

$$\begin{array}{ccc} V_{ta} & \xrightarrow{V_a} & V_{ha} \\ \downarrow{\scriptstyle \phi_{ta}} & & \downarrow{\scriptstyle \phi_{ha}} \\ W_{ta} & \xrightarrow[W_a]{} & W_{ha} \end{array}$$

commutes for every arrow $a \in Q_1$. That is, $W_a \phi_{ta} = \phi_{ha} V_a$ for all $a \in Q_1$.

For a quiver Q and a field K we can form the *category* $\mathrm{Rep}_K(Q)$ whose objects are representations of Q with the morphisms as defined above.

A morphism $\phi : V \to W$ is an *isomorphism* if ϕ_x is invertible for every $x \in Q_0$. One naturally wants to classify all representations of a given quiver Q up to isomorphism.

Example 4. *Consider Example 1. For a linear map $V_a : V_1 \to V_2$ we can always choose bases in V_1 and in V_2 in which V_a is given by the block matrix*

$$\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

where r is the rank of A and I_r is the $r \times r$ identity matrix. Two representations $V_a : V_1 \to V_2$ and $W_a : W_1 \to W_2$ are isomorphic if and only if $\dim V_1 = \dim W_1$, $\dim V_2 = \dim W_2$, and V_a and W_a have the same rank.

Example 5. *Consider Example 2. Assume that the base field K is algebraically closed. If $V_a : V_1 \to V_1$ is an endomorphism of the finite-dimensional K-vector space V_1, then for some choice of basis in V_1, the matrix of V_a has the form*

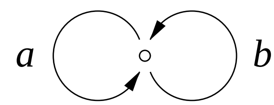
$$(1) \qquad \begin{pmatrix} J_{n_1,\lambda_1} & 0 & \cdots & 0 \\ 0 & J_{n_2,\lambda_2} & & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & J_{n_r,\lambda_r} \end{pmatrix},$$

where $J_{n,\lambda}$ denotes the $n \times n$ Jordan block

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & \varnothing & \\ & & \ddots & \ddots & \\ & \varnothing & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}.$$

The matrix (1) is the well-known Jordan normal form of V_a. It is unique up to permutation of the blocks. Two representations, $V_a : V_1 \to V_1$ and $W_a : W_1 \to W_1$, of the loop quiver are isomorphic if and only if V_a and W_a have the same Jordan normal form.

As we have seen, the classifications of representations of the quivers in Examples 1 and 2 correspond to well-known problems in linear algebra. For more complicated quivers, the classification problem leads to more involved linear algebra problems. For example, for the double loop quiver



we have to classify all pairs of matrices (V_a, V_b) up to *simultaneous* conjugation, a notoriously difficult problem. The classification problem for Example 3 is equally hard. Yet there are many quivers for which the classification problem has been solved.

Indecomposable Representations

If V and W are two representations of the same quiver Q, we define their *direct sum* $V \oplus W$ by

$$(V \oplus W)_x := V_x \oplus W_x$$

for all $x \in Q_0$, and

$$(V \oplus W)_a := \begin{pmatrix} V_a & 0 \\ 0 & W_a \end{pmatrix} : V_{ta} \oplus W_{ta} \to V_{ha} \oplus W_{ha}$$

for all $a \in Q_1$.

We say that V is a *trivial representation* if $V_x = 0$ for all $x \in Q_0$. If V is isomorphic to a direct sum $W \oplus Z$, where W and Z are nontrivial representations, then V is called *decomposable*. Otherwise V is called *indecomposable*. Every representation has a unique decomposition into indecomposable representations (up to isomorphism and permutation of components). The classification problem reduces to classifying the indecomposable representations.

Example 6. *Let us go back to Examples 1 and 4. There are 3 indecomposable representations A, B, C, namely*

$$A : K \to 0, \quad B : 0 \to K, \quad C : K \xrightarrow{1} K.$$

Any representation V is isomorphic to

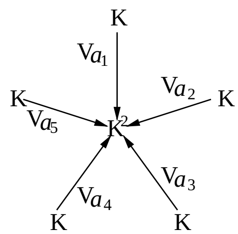
$$V \cong A^{d_1 - r} \oplus B^{d_2 - r} \oplus C^r$$

where $d_1 = \dim V_1$, $d_2 = \dim V_2$ and $r = \operatorname{rank} V_a$.

Example 7. *Consider again Examples 2 and 5. Indecomposable representations correspond to the Jordan blocks. The Jordan normal form shows how a representation decomposes into indecomposables.*

Although there are infinitely many indecomposable representations, they can still be parametrized by a discrete parameter n and a continuous parameter λ.

Example 8. *In Example 3, one can identify a 2-dimensional family of pairwise nonisomorphic indecomposable representations, namely,*



where $V_{a_1}, \ldots, V_{a_5}$ *are given by the matrices*

$$\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\\lambda\end{pmatrix}, \begin{pmatrix}1\\\mu\end{pmatrix},$$

respectively, with $\lambda, \mu \in K$.

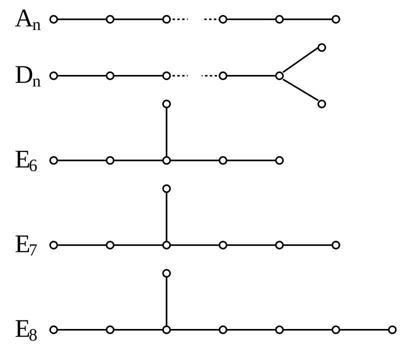
Furthermore, there exist other families of indecomposables for this particular star quiver, where the number of parameters of the family is arbitrarily large. In this example, describing explicitly the set of indecomposable representations is essentially an impossible task.

Theorems of Gabriel and Kac

We have observed different behavior of indecomposables for various quivers. If a quiver has only finitely many indecomposable representations, it is called a quiver *of finite type*. If there are infinitely many indecomposables, but they appear in families of dimension at most 1, then the quiver is called *of tame type*.[2] If the representation theory of the quiver is at least as complicated as the representation theory of the double loop quiver, then the quiver is called *of wild type*. These definitions given here are imprecise but hopefully convey the right intuition. The precise definitions of tame and wild type are omitted. It is known that every quiver is either of finite type, tame, or wild. We will later see that such a trichotomy is true in a more general setting.

Forgetting the orientations of the arrows yields the *underlying undirected graph* of a quiver. The following amazing theorem is due to Gabriel (see [8], [13]).

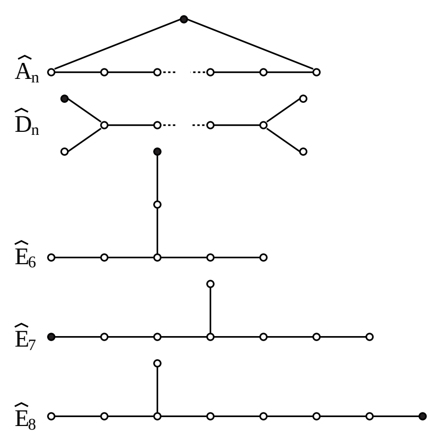
Theorem 9 *[Gabriel's Theorem, part 1]. A quiver is of finite type if and only if the underlying undirected graph is a union of Dynkin graphs of type A, D, or E, shown below:*



The Dynkin graphs play an important role in the classification of simple Lie algebras, of finite crystallographic root systems and Coxeter groups, and other objects of "finite type".

For quivers of tame type, a similar description exists, namely:

Theorem 10 *([5], [14]). A quiver Q which is not of finite type is of tame type if and only if the underlying directed graph is a union of Dynkin graphs and extended Dynkin graphs of type* $\hat{A}$, $\hat{D}$, *or* $\hat{E}$, *shown below:*



Gabriel proved a stronger statement for quivers of finite type:

Theorem 11 *[Gabriel's Theorem, part 2]. The indecomposable representations are in one-to-one correspondence with the positive roots of the corresponding root system. For a Dynkin quiver Q, the dimension vectors of indecomposable representations do not depend on the orientation of the arrows in Q.*

Amazingly, this result is just the tip of an iceberg. Define the *Euler form* (or *Ringel form*) of a

[2] *In some papers, the definition of tame type includes finite type.*

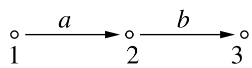
quiver Q to be the asymmetric bilinear form on $\mathbb{Z}^{Q_0}$ given by

$$\langle \alpha, \beta \rangle = \sum_{x \in Q_0} \alpha(x)\beta(x) - \sum_{a \in Q_1} \alpha(ta)\beta(ha).$$

The Euler form is represented in the coordinate basis of $\mathbb{Z}^{Q_0}$ by the matrix $E = (b_{i,j})$ where the

$$b_{i,j} = \delta_{i,j} - \#\{\, a \in Q_1 \mid ta = i,\ ha = j\},$$

where $\delta_{i,j}$ is the Kronecker delta symbol. One also defines a symmetric bilinear form



$$(\alpha, \beta) := \langle \alpha, \beta \rangle + \langle \beta, \alpha \rangle,$$

called *the Cartan form* of the quiver Q. The Cartan form does not depend on the orientation of the arrows in Q.

Example 12. *For the quiver*

the Euler matrix is

$$E = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$

and the Cartan matrix is

$$C = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}.$$

The *Tits form* q of Q is defined by

$$q(\alpha) = \langle \alpha, \alpha \rangle = \tfrac{1}{2}(\alpha, \alpha).$$

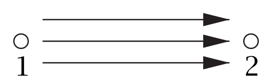
The number of continuous parameters for α-dimensional ($\alpha \neq 0$) representations is known to be at least $1 - q(\alpha)$. The Tits form plays an integral role in Gabriel's theorem. For a quiver of finite type and a nonzero dimension vector α, there are only finitely many representations up to isomorphism, so $q(\alpha) \geq 1$. From this one can prove that the Cartan form is positive definite and that the underlying undirected graph is a union of Dynkin diagrams. One can also show that a dimension vector α is a positive root if and only if $q(\alpha) = 1$.

About the same time at which Gabriel proved his theorem, Kac and Moody came up with a generalization of root systems and corresponding Lie algebras for Cartan matrices of arbitrary quivers. Kac proved in 1980 the following result (see [9]).

Theorem 13 *[Kac's Theorem]. For an arbitrary quiver Q, the set of dimension vectors of indecomposable representations of Q does not depend on the orientation of arrows in Q. The dimension vectors of indecomposable representations correspond to positive roots of the corresponding root system.*

In the theory of Kac-Moody algebras one distinguishes between *real roots* and *imaginary roots*. In Theorem 13, real roots correspond to dimension vectors for which there is exactly one indecomposable representation, while imaginary roots correspond to dimension vectors for which there are families of indecomposable representations. If a positive root α is real, then $q(\alpha) = 1$. If it is imaginary, then $q(\alpha) \leq 0$.

Example 14. *The real roots for the wild quiver*



are

$$\begin{array}{l} (1,0),(3,1),(8,3),(21,8),\ldots \\ (0,1),(1,3),(3,8),(8,21),\ldots \end{array}$$

(pairs of consecutive odd Fibonacci numbers). The imaginary roots are all $(p,q) \in \mathbb{N}^2$ with

$$\frac{3-\sqrt{5}}{2} < \frac{p}{q} < \frac{3+\sqrt{5}}{2}.$$

The connections with the theory of Lie algebras and algebraic groups can be developed much further. Ringel showed how to construct the upper triangular part of the enveloping algebra of a simple Lie algebra from the representations of the corresponding Dynkin quiver Q, using the Hall algebra associated to Q ([15]). The connections between quiver representations and canonical bases of quantum groups is an active area of current research.

Canonical Decompositions

Kac's theorem describes the dimension vectors in which indecomposable representations appear. However, this theorem does not tell us how to construct indecomposable representations. One might think that a "generic" representation of dimension α is indecomposable if α is a root. This is not the case. Because the classification of (indecomposable) representations is no longer feasible, we will set ourselves more modest goals. We will ask ourselves the following questions:

If we fix a dimension vector α and we choose all the linear maps at random, when will such a representation be indecomposable? When will such a representation be rigid? (This means: for which representations does every small enough perturbation of the linear maps result in an isomorphic representation?)

We say that a *general* representation of dimension α is indecomposable if there is a nontrivial polynomial equation in the entries of the matrices such that every decomposable representation of dimension α satisfies the polynomial equation.

Even for the loop quiver from Example 2 we see that in dimension vector (n) a representation can be indecomposable only if all its eigenvalues are the same. One might ask a different question: how does a general representation decompose? Kac showed that if V is a "sufficiently general" representation with dimension vector α, then the dimension vectors of the direct summands will not depend on V. This general decomposition of the dimension vector of α into the dimension vectors of indecomposable summands is called the *canonical decomposition of* α. This notion depends on the orientation of arrows in Q. We write

$$\alpha = \alpha_1 \oplus \ldots \oplus \alpha_r$$

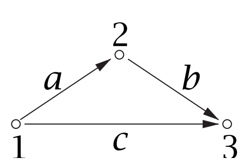
if a general representation of dimension vector α has r indecomposable summands of dimension vectors $\alpha_1, \ldots, \alpha_r$. If a general representation of dimension α is indecomposable then the canonical decomposition of α is just α itself. In this case, α is called a *Schur root*.

Example 15. *Take the loop quiver from Example 2. The dimension vector $\alpha = (n)$ is an imaginary root because there is a 1-dimensional family of indecomposables in this dimension. The canonical decomposition of α is*

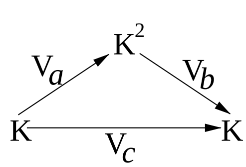
$$(n) = \underbrace{(1) \oplus (1) \oplus \cdots \oplus (1)}_{n} = (1)^{\oplus n}$$

because a general endomorphism has distinct eigenvalues and thus decomposes into Jordan blocks of size one.

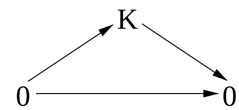
Example 16. *Consider the quiver*



A general representation of dimension $(1, 2, 1)$ is of the form



This representation is indecomposable if V_a is injective, V_b is surjective, V_c is an isomorphism, and $V_bV_a = 0$. The dimension vector $(1, 2, 1)$ is a real root (up to isomorphism there is only one indecomposable representation). For a general representation of dimension $(1, 2, 1)$, however, the composition V_bV_a will be nonzero, and an indecomposable summand



will split off. Thus $(1, 2, 1)$ is not a Schur root.

The canonical decomposition is homogeneous in the following way. If

$$\alpha = \alpha_1 \oplus \ldots \oplus \alpha_r$$

is the canonical decomposition for some dimension vector, then

$$n\alpha = [n\alpha_1] \oplus \ldots \oplus [n\alpha_r], \tag{2}$$

where $[n\alpha]$ denotes $n\alpha$ if $\langle \alpha, \alpha \rangle < 0$ and $\alpha^{\oplus n}$ if $\langle \alpha, \alpha \rangle \geq 0$.

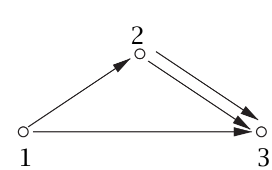
We now get back to the questions from the beginning of this section. A dimension vector α is rigid if every summand in the canonical decomposition of α is a real Schur root. An efficient combinatorial algorithm to compute the canonical decomposition of a dimension vector was given in [4] (a similar algorithm is given in [17]). Using this algorithm it is possible to check whether a given dimension vector is a Schur root or a rigid dimension vector. It is unlikely that an easy explicit description of Schur roots or dimension vectors exists, given the complex nature of these notions revealed in the next section.

An Example

We will discuss an example to visualize the notions of real and imaginary roots and Schur roots.

The sets of dimension vectors for which general representation is indecomposable have a very complicated structure, as we will see in the example below. For a quiver with three vertices, we can graph dimension vectors in the projective plane to get a two-dimensional picture. A dimension vector (x, y, z) will be drawn as the projective point $[x : y : z]$ in $\mathbb{P}^2$. This makes sense because the canonical decomposition is essentially homogeneous by (2). All dimension vectors will be contained in the triangle $[x : y : z]$, $x, y, z \geq 0$.

Let Q be the quiver



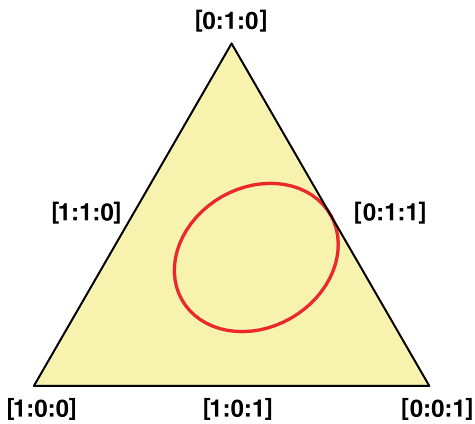


Figure 1.

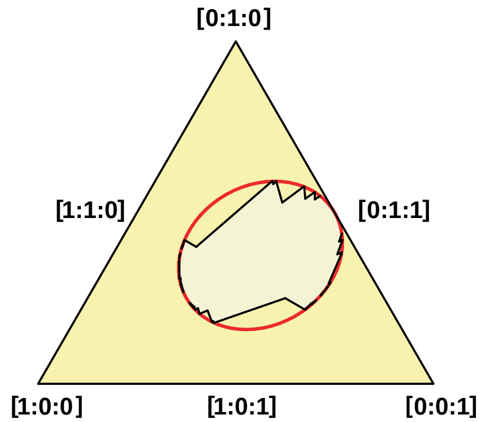


Figure 2.

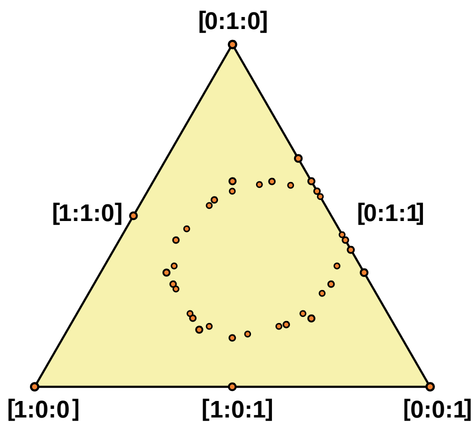


Figure 3.

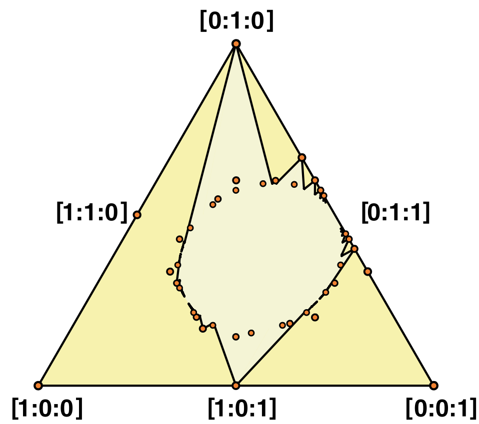


Figure 4.

For this particular quiver, the imaginary positive roots are exactly all dimension vectors α for which $q(\alpha) \leq 0$. The quadric $q(\alpha) = 0$ is plotted in Figure 1. This quadric and its interior correspond to the imaginary roots.

The imaginary Schur roots are the dimension vectors inside the nonconvex fractal-like polygon shown in Figure 2. Since the polygon is properly contained inside the quadric in Figure 1, we see that there exist imaginary roots that are not imaginary Schur roots.

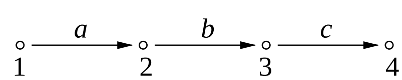
In figure 3 we plotted some real roots for this quiver. The dimension vectors α for which a general representation is rigid are those that lie outside the fractal-like polygon in the Figure 4. A real root α is a real Schur root if and only if α is a rigid dimension vector. We see that some of the real roots in Figure 3 lie inside the polygon in Figure 4. This shows that some real roots are not real Schur roots.

Representation Theory of Finite-Dimensional Algebras

There is a close connection between quivers and the representation theory of finite-dimensional algebras. In the last few decades there has been an enormous progress in the area of finite-dimensional algebras. Because of space limitations we will not be able to do justice to this subject. We will just give a glimpse of this area and its connection to quivers.

A path in a quiver Q is a sequence $a_1a_2\cdots a_r$ of arrows in Q_1 with $ta_i = ha_{i+1}$ for $i = 1, 2, \ldots, r-1$. We also define a trivial path e_x with $te_x = he_x = x$ for each vertex $x \in Q_0$. The *path algebra* of KQ is the vector space spanned by all paths in Q. The algebra structure of KQ is given by the concatenation of paths. There is a natural bijection between representations of the quiver Q and (left)-KQ-modules.

Example 17. *Consider the quiver*



For every i, j with $1 \leq i \leq j \leq 4$ there is a unique path from i to j. Identify this path from i to j with the matrix $E_{j,i}$ having a 1 in row j and column i and 0 everywhere else. Using this identification, we see that the path algebra for this quiver is isomorphic to the set of 4×4 lower triangular matrices.

If A is a finite-dimensional algebra over the complex numbers $\mathbb{C}$, then the category of representations of the algebra A is equivalent to the category of representations of the algebra KQ/I for some quiver Q and some two-sided ideal I of KQ. This is the reason why quivers play a central role in the theory of finite-dimensional algebras and their modules.

One can extend the notions of *finite, tame,* and *wild* type for finite-dimensional algebras. An important result for quivers with relations is Drozd's Theorem, which states that every finite-dimensional algebra is either finite type, tame, or wild (see [3], [7]). These possibilities are mutually exclusive.

Even though the classification of indecomposable representations of wild algebras is an almost impossible task, King showed that it is possible to construct nice moduli spaces for the representations using geometric invariant theory [12]. These moduli spaces do not parameterize all representations, but only representations that are a direct sum of indecomposable representations satisfying a certain stability condition.

Another direction in representation theory of quivers started with Auslander and Reiten's application of homological methods. They introduced what is nowadays called the Auslander-Reiten transform of a representation of a finite-dimensional algebra. For representations of a quiver Q without relations or oriented cycles, the Auslander-Reiten transform induces a map of dimension vectors. If V is a representation of dimension α for which the Auslander-Reiten transform is defined, then its transform has dimension $\tau(\alpha)$ where $\tau(\alpha) \in \mathbb{Z}^{Q_0}$ is the unique integer vector satisfying

$$\langle \tau(\alpha), \beta \rangle = -\langle \beta, \alpha \rangle$$

for all dimension vectors β. The homological properties of the Auslander-Reiten transform imply that if α is a real root/imaginary root/real Schur root/imaginary Schur root/rigid dimension vector, then so is $\tau(\alpha)$.

The Auslander-Reiten transform is visible in the figures of the previous section. In that case, the map τ is given by

$$\begin{pmatrix} 3 & 5 & -3 \\ 3 & 3 & -2 \\ 1 & 2 & -1 \end{pmatrix}.$$

Note that for example $\tau(0,1,1) = (2,1,1)$ and $\tau(1,0,1) = (0,1,0)$. Using the homological properties of the Auslander-Reiten transform, we have discovered the map τ that leaves the quadric in Figure 1, the infinite polygon in Figure 2, the real roots[3] (Figure 3), and the infinite polygon in Figure 4 invariant.

[3] *The set of positive real roots is not completely invariant under τ. If R is the set of positive real roots, then each of $\tau(R) \setminus R$ and $R \setminus \tau(R)$ consists of only three vectors.*

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