Book Review

… a small idea of what it is I do all day…

Introductions to Mathematics

Günter M. Ziegler

What Is Mathematics? An Elementary Approach to Ideas and Methods
Richard Courant and Herbert Robbins
Second edition, revised by Ian Stewart
Oxford University Press, 1996
Paperback, 592 pages, $21.50
ISBN 0-19-510519-2
The Heart of Mathematics: An Invitation to Effective Thinking
Edward B. Burger and Michael Starbird
Key College Publishing (Springer-Verlag), 2000
Hardcover, 646 pages, $69.95
ISBN 0-555953-407-9
Mathematics: A Very Short Introduction
Timothy Gowers
Oxford University Press, 2002
Paperback, 144 pages, $9.95
ISBN 0-19-285361-9
1089 and All That. A Journey into Mathematics
David Acheson
Oxford University Press, 2002
Hardcover, 178 pages, £9.95
ISBN 0-19-851623-1

What Is Mathematics?

Here are some questions for you:

1. What is mathematics?
2. What are you doing when you are "doing mathematics"?
3. Mathematical research—how does that happen? Can it be planned? Can it be cast into "projects"?
4. What are the most important problems in mathematics? And why don't all mathematicians work on these?
5. What does it mean to a mathematician that something is *proved*? And what does it mean if he *believes a proof*?
6. How and why is mathematics relevant for "the real world"?
7. Why is mathematics so unreasonably effective in some applications?
8. Why is mathematics so hard to understand? And why is most of it so hard to explain?

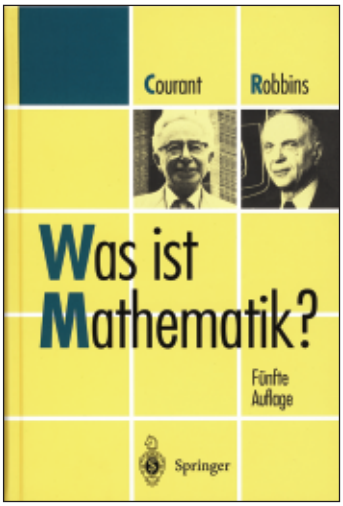
All these are valid questions. There are classical answers: G. H. Hardy's *A Mathematician's Apology* probably contains the most famous and the most controversial ones. But none of them are eternal or universally valid. As mathematics and the nature of mathematics develop and change, we have to come up with new explanations again and again. One may also try to come up with answers that are not even intended to be universally valid, with personal views of the world of mathematics. I believe that these are equally useful and important. So, what is mathematics *to you*?

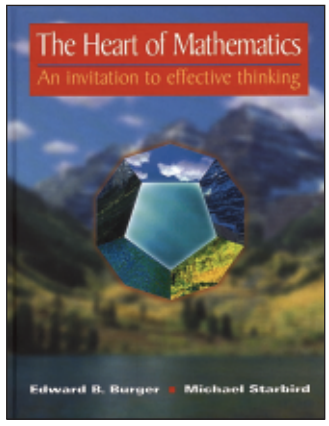
Suggested Answers

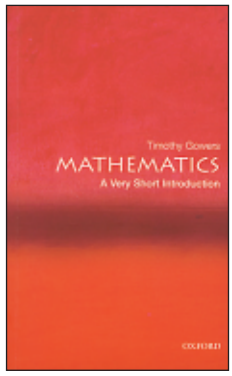
One reason why you should look for, and provide, your own answers to such questions is that you won't like many of the answers that you'll get from others.

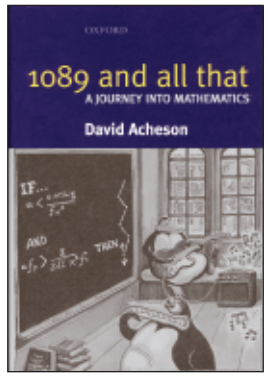
For example: Why is it that these days we start with *applications* whenever we try to present mathematics to the public? Perhaps this is the easiest and the most effective way, or the most convincing? The one that even politicians might understand? My own experience certainly supports this impression. I can't explain the high-dimensional geometry I do to nonmathematician friends at night at a bar, but I can tell them about mathematicians'

Günter M. Ziegler is professor of mathematics at Technische Universität Berlin. His email address is ziegler@math.tu-berlin.de.









work on bus schedules and on hyperthermia cancer therapy (to mention two Berlin examples), and they'll be suitably impressed. However, although applications answers may be interesting and easy and effective, they shouldn't be the only ones!

Another example: What kind of pictures of "what mathematics is like" do your kids bring home from school? Do these pictures have anything to do with what we believe they should be? Aren't their views of mathematics very lopsided and incomplete, at best? Do they represent what mathematics "really is about"? Do kids experience the power of mathematics, the virtues of precision and abstraction, mathematical creativity and mathematical ideas? Do they learn about this in a way that we think is fair, colorful, multifaceted, exciting, and inviting? Do they get to see mathematics as we would view it? How do we view it?

Mathematics According to Courant

What Is Mathematics? is the title of a classic book by Richard Courant and Herbert Robbins from 1941, published by Oxford University Press, with successful translations into German, Italian, and Russian. Apparently Courant was quite hesitant about the title and thought that *Mathematical Discussions of Basic Elementary Problems for the General Public* might be more accurate, but also "a little bit boring."

Constance Reid tells the story that at a dinner in Princeton Courant talked to Thomas Mann, who told him about one of his books, a little novel whose German title was *Lotte in Weimar*. Mann thought that the same title might work in English, whereas his publisher Alfred Knopf (or rather Knopf's wife) had suggested *The Beloved Returns*, saying that *Lotte in Weimar* would sell 10,000 or perhaps 20,000 copies, but *The Beloved Returns* might sell 100,000—with the corresponding authors' royalties. So Thomas Mann went for *The Beloved Returns*, and Richard Courant went for *What Is Mathematics?* Courant sold more than 100,000 copies. I don't know about Mann's book.

Courant was perhaps one of the most influential "applied mathematicians" of the twentieth century. But there's no "applied math" in his book! However, there is a lot of mathematics, just very little *about* mathematics. Courant and Robbins' answer to his title question is to develop and explain mathematics, a wealth of very classical and fundamental topics: numbers and number theory, geometry, topology, calculus. The last sentence of the introduction reads

> For scholars and laymen alike it is not philosophy but active experience in mathematics itself that alone can answer the question: What is mathematics?

New Answers

What is mathematics? Is it changing? Of course it is, and so we have to ask for, and attempt to give, *new* answers to the old question.

The Heart of Mathematics: An Invitation to Effective Thinking is the title of one such answer, a book by Edward B. Burger (Williams College) and Michael Starbird (University of Texas at Austin). It is a large volume (650 pages in the first edition), with ample four-color illustrations throughout, and a bit of arrogance in the title. What a contrast to Courant's book, in style, but also in contents. This is a textbook for a "math appreciation" undergraduate class, but also fun reading and viewing for anyone else. The choice of topics is guided by what is or should be interesting to an audience of non-science majors. So some classical topics appear; after a lot of introductory drumming the book starts with the pigeon-hole principle and Fibonacci numbers. But this is not part of a systematic development. There are rope tricks, games, aperiodic tilings, fractals, lots of "modern mathematics", all of this embedded into lots of motivational talk, "fun and games", "mindscapes", "creating new ideas", "invitations to further thought", and "lessons for life". It's well done, I think it is good to have, it is

fun, but it is not my style. At forty-one, I may be too old or too old-fashioned for this.

Personally, I prefer the small and modest attempts to answer the old question. One recent such attempt is *Mathematics: A Very Short Introduction* by Tim Gowers (of 1998 Fields Medal fame). It appeared in an Oxford University Press series of "Very Short Introductions" that treats an extensive list of topics concerning nearly everything under the sun, such as religion, philosophy, and history, but also animal rights, cryptography, evolution, Heidegger, Hinduism, and linguistics. There's even "Schizophrenia: A Very Short Introduction".

An Author Needs to Be Convinced

Let's just imagine one of the friendly Oxford University Press mathematics editors knocking at Professor Gowers's office door. She would introduce herself, and say "Well, you know, we have this little book series ... it is really quite successful ... this, for example, is the little volume on atheism ... it has just appeared ... couldn't you, for example, write one for us ... about mathematics ...??" And Professor Gowers is quite thoughtful and a bit skeptical, and he doesn't want to say no right away, because somehow he *is* intrigued by the challenge, so he just promises "... to think about it ..." And then it comes as a considerable surprise (first to him, then to his editor) that he does get started on the impossible task, takes on the challenge. And *he succeeds*, with style.

Mathematics According to Gowers

The result is quite British, serious but not without humor. The preface sets the stage:

> Very little prior knowledge is needed to read this book [...] but I do presuppose some interest on the part of the reader rather than trying to drum it up myself. For this reason I have done without anecdotes, cartoons, exclamation marks, jokey chapter titles, or pictures of the Mandelbrot set. I have also avoided topics such as chaos theory and Gödel's theorem, which have a hold on the public imagination out of proportion to their impact on current mathematical research, and which are in any case well treated in many other books.

And then he gets going: There's not much space—140 *small* pages. No space to *explain* large amounts of mathematics. But Gowers gives a captivating, interesting, and quite personal[1] introduction into some mathematical questions, which surprisingly fast get surprisingly close to the "heart of mathematics", without having to do any difficult math. He speaks clearly and concretely about the role of models and about abstractions, concluding "Once one has learned to think abstractly, it can be exhilarating, a bit like suddenly being able to ride a bicycle without having to worry about keeping one's balance." He talks about limits and infinity and about dimension, and at this point even a fractal curve comes up, including a picture! And he explains why most mathematicians don't worry about foundational questions such as "whether numbers exist" or about "infinity".

Gowers' last chapter, "Some Frequently Asked Questions", treats mathematics as a human endeavor. Here's his list of questions:

1. Is it true that mathematicians are past it by the time they are thirty?
2. Why are there so few women mathematicians?
3. Do mathematics and music go together?
4. Why do so many people positively dislike mathematics?
5. Do mathematicians use computers in their work?
6. How is research in mathematics possible?
7. Are famous mathematical problems ever solved by amateurs?
8. Why do mathematicians refer to some theorems and proofs as beautiful?

Of course I won't give you his answers here. You should look at his volume: He certainly does not give "the only possible correct answers", but convincing, modest, and thoughtful ones. For example, on Question 4 he does (of course!) talk about instruction at schools, and he thinks that there's need and room for improvement, but also he says "I do not advocate any revolutionary change—mathematics has suffered from too many of them already—but a small change in emphasis could pay dividends. For example [...]". On Question 8, he thinks that "from an aesthetic point of view, a mathematician is more anonymous than an artist. While we may greatly admire a mathematician who discovers a beautiful proof, the human story behind the discovery eventually fades away." Does it? I believe that it's our duty to also record the human story, the story about what we all do, the stories about those who solve the big problems—and about those who put decades into the quest for such solutions.

Mathematics According to Acheson

David Acheson's *1089 and All That. A Journey into Mathematics* is a little volume from the same publisher as Gowers's book. It appeared in the same year, in the same small format—but what a contrast it is otherwise! This contains all the "anecdotes, cartoons, exclamation marks, jokey chapter titles" that Gowers wouldn't do. It's an amusing and entertaining roller-coaster ride into the world of mathematics, as seen by David Acheson. It is a

[1] *The title of this review quotes the last few words from the preface.*

lively, funny, multifaceted view, with lots of pictures, drawings, cartoons: the ride starts with a magic trick based on numbers (whose result is 1089) from *I-SPY* magazine that intrigued Acheson as a boy. In the end, it leads up to an elementary discussion of differential equations, in which he now, perhaps not quite seriously, sees answers to the "mystery of life". But in any case differential equations provide useful models for the real world, and they do have applications. Probably *1089 and All That* is not an *important* book in any sense, but a fun piece to look into on a rainy afternoon.

Mathematics According to You!

If you are a mathematician, and not one of the very introverted kind, then I think you ought to give your own answers. You needn't think about it as starting to write a book. Just get a little notebook, perhaps the size in which we wrote our diaries in the good old times when people like us would still write *by hand*. To begin, put down your own list of questions that you think need good answers; you might start with "What is mathematics?"

Whatever you want to share with people, make it available. Some colleagues have a few pages of notes, views, or advice for students on their home pages. The most outspoken example I know is Doron Zeilberger's famous, wonderfully outrageous "Opinions" page.[2] Why are mathematician's professional home pages so impersonal? I don't mean that I want to see more baby photos—I'd want to see more thoughts about what you are doing, what the mathematics means to you, where the challenges are, and why this is interesting and important to you and to all of us.

[2] *"Dr. Z's opinions,"* `http://www.math.rutgers.edu/~zeilberg/OPINIONS.html`.