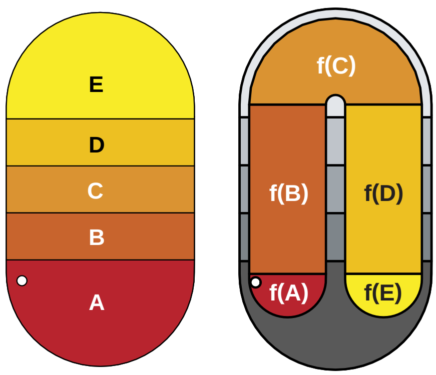
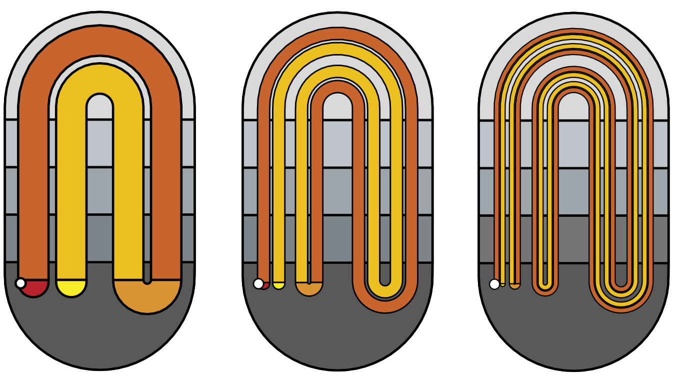
Picturing the Horseshoe Map



Smale's horseshoe map takes a disk into itself. Points in A and E are mapped into A, points in C into E.



Upon iteration, any point that finds itself in A, C, or E eventually winds up in A, converging to an attracting fixed point in that region. The interesting dynamics happens to points that never arrive in A, C, E. This is the intersection of all the sets

$$\Omega_{0,n} = \{x \mid f^k(x) \in \mathrm{B} \cup \mathrm{D} \text{ for } 0 \le k \le n\},$$

defined recursively by

$$\Omega_{0,0} = \mathrm{B} \cup \mathrm{D}$$
$$\Omega_{0,n+1} = \Omega_{0,n} \cap f^{-(n+1)}(\mathrm{B} \cup \mathrm{D}).$$

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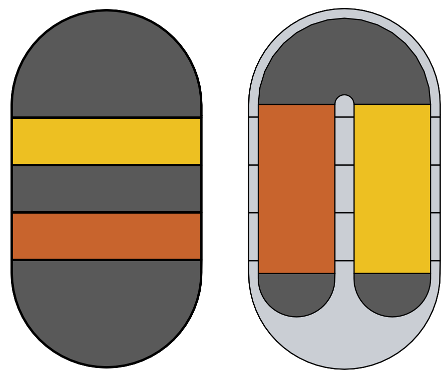
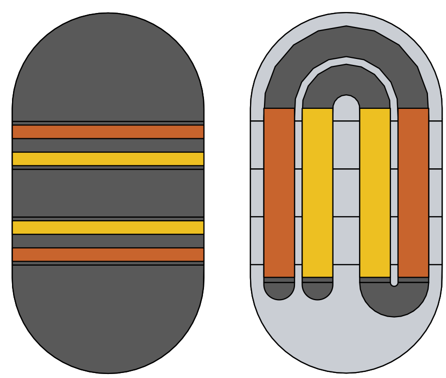
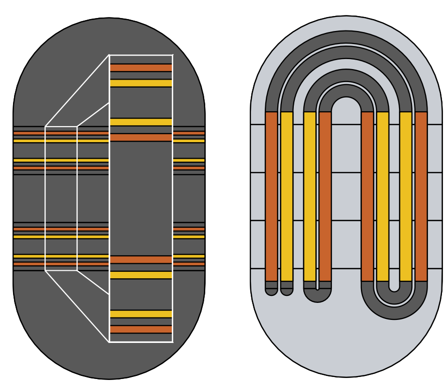


$\Omega_{0,0} = \mathrm{B} \cup \mathrm{D}$ $\quad f(\Omega_{0,0})$



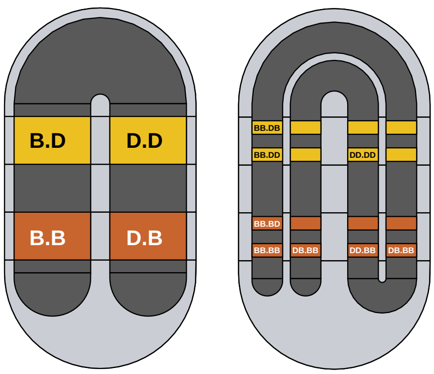
$\Omega_{0,1}$ $\quad f^2(\Omega_{0,1})$



$\Omega_{0,2}$ $\quad f^3(\Omega_{0,2})$

These converge to a collection $\Omega_{0,\infty}$ of horizontal bands distributed vertically like Cantor dust.

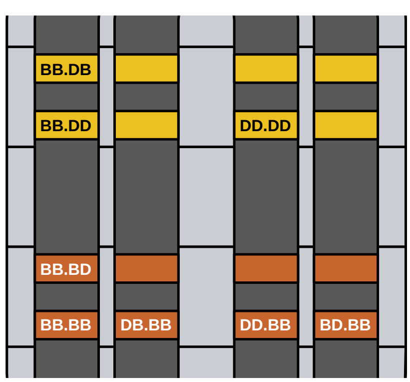
Points in $\Omega_{0,\infty}$, as iterations of f are applied to them, approach a set on which f is invertible. This is the set $\Omega_{-\infty,\infty}$ of points x for which all $f^n(x)$ lie in $\mathsf{B} \cup \mathsf{D}$. The condition that $f^{-n}(x)$ lie in $\mathsf{B} \cup \mathsf{D}$ is equivalent to the condition that x lie in $f^n(\mathsf{B} \cup \mathsf{D})$. This set is approximated by sets $\Omega_{-m,n}$ with m and n large, and amounts to a two-dimensional Cantor dust.



$\Omega_{-1,0}$ $\Omega_{-2,1}$

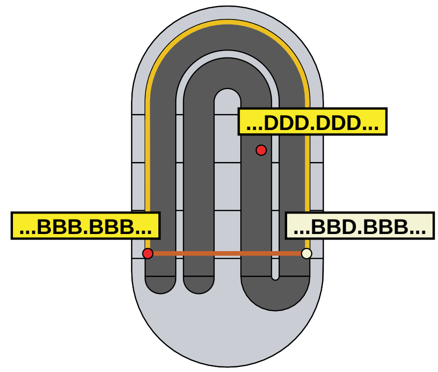
Because the map f is an affine transformation on B and D, these sets are easy to calculate, and it is easy to see that the connected components converge to isolated points. Each one of the connected sets $\Omega_{-m,n}$ can be labeled by a finite string $s_{-m} \ldots s_{-1}.s_0 s_1 \ldots s_n$ of characters B and D where $s_k = \mathsf{B}$ if $f^k(\Omega) \subseteq \mathsf{B}$, otherwise D. Thus B.D is the set of points x with x in D, $f^{-1}(x)$ in B. The points to which the sets converge are indexed by infinite strings (s_k) for k in $\mathbb{Z}$. The limit set is stable under f, which acts as a left shift. The image of BB.DB under f is BBD.B.

The coordinate succession is semi-inverted:



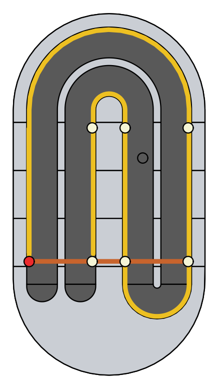
Coordinates $x.y$

There are exactly two points fixed by f, namely $\overline{\mathsf{B}} = \ldots\mathsf{BB.BB}\ldots$ and $\overline{\mathsf{D}} = \ldots\mathsf{DD.DD}\ldots$, which are both hyperbolic—attracting horizontally and repelling vertically. The points with an infinite string $\ldots\mathsf{BBB}\ldots$ at the right end are those attracted to the first, those with an infinite terminal string $\ldots\mathsf{DDD}\ldots$ to the second. Points $\ldots\mathsf{BB}\ldots\mathsf{BB}\ldots$ with an infinite string of B at both right and left other than $\overline{\mathsf{B}}$ itself are **homoclinic** points, attracted to $\overline{\mathsf{B}}$ upon iteration by both f and f^{-1}.



Hyperbolic fixed points and a homoclinic point

The simplest homoclinic point is $\ldots\mathsf{BBD.BB}\ldots$, which lies on both the horizontal line through the hyperbolic point $\mathsf{P} = \ldots\mathsf{BBB.BB}\ldots$ and the image under iterations of f applied to a small vertical segment through P.



More homoclinic points

But the homoclinic points are in fact dense in $\Omega_{-\infty,\infty}$, as the picture above barely suggests. In fact, as it I hope it also suggests, the horseshoe map and the network of homoclinic points in very general dynamical systems are intimately related. This was Smale's original insight on the beach at Rio, mentioned in Shub's article in this issue.

—*Bill Casselman*