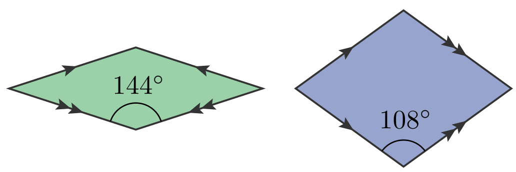
Up and Down the Tiles

David Austin

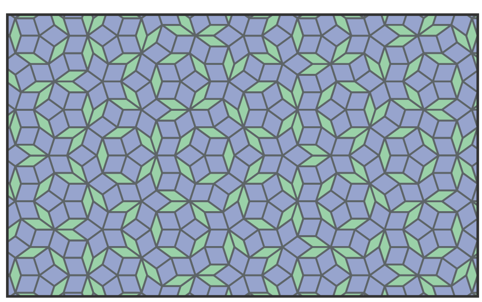
United States Patent 4,133,152, filed in 1975 by Roger Penrose, describes a game played with two rhombs



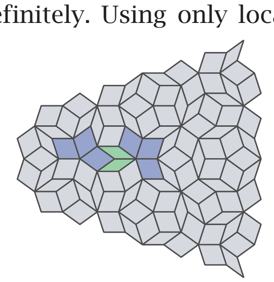
and asserts that with them

> *…one can play a form of solitaire. A large supply of pieces is presented…. One may simply play with the pieces and cover as large an area as possible* [matching the arrows on the edges of adjacent rhombs], *producing many intriguing and ever-varying patterns in the process.* [8]

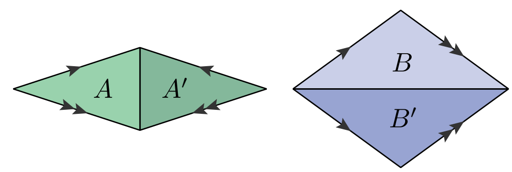
That is, the object of the game is to construct an arbitrarily large—ideally, indefinitely extensible—tiling of the playing area.



Playing this game (see [9], for example) requires some thought. For example, we may begin with the colored tiles shown below, but unless we next lay down the gray tiles, our configuration cannot be extended indefinitely. Using only local techniques, such as matching the arrows, will often lead to patterns with no extension (see [6]).

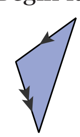


De Bruijn [1] described an ingenious method for constructing Penrose tilings by working with half-rhombs.

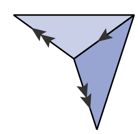


Write $A \uparrow B'$ to indicate that A and B' are joined across an edge with a single arrow. Across an arrowed edge on a half-rhomb, only two possible half-rhombs may be joined.

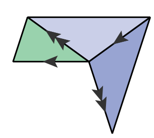
Begin laying down half-rhombs like this:



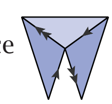
Start with B'.



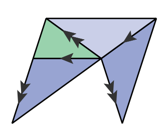
Add a $B' \uparrow B$ connection.



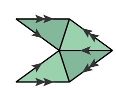
Next comes $B \upuparrows A$, since



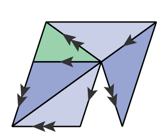
may not be extended.



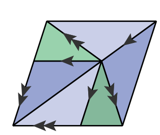
Add $A \uparrow B'$, since



may not be extended.



Complete the half-rhomb B' by adding B.



Add A' to form a thick rhomb whose linear dimensions are scaled by the golden ratio $\tau = (1 + \sqrt{5})/2$.

Beginning instead with $B' \uparrow A$ leads to a thin rhomb scaled by τ.

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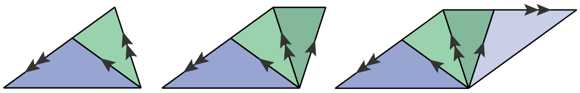
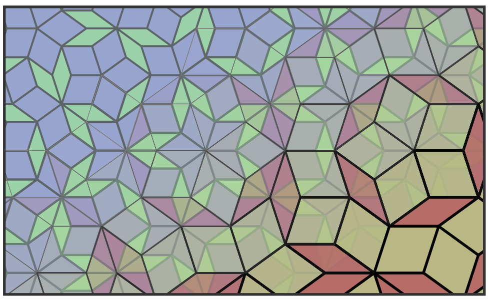
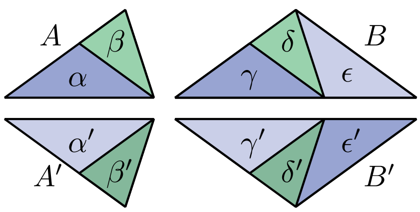


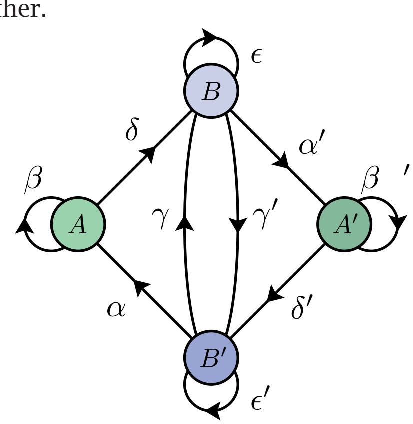
In fact, every half-rhomb in a tiling fits uniquely into one of these two chains, and so the half-rhombs may be grouped in exactly one way to form a second tiling, called the *inflated* tiling, constructed with rhombs scaled by τ.



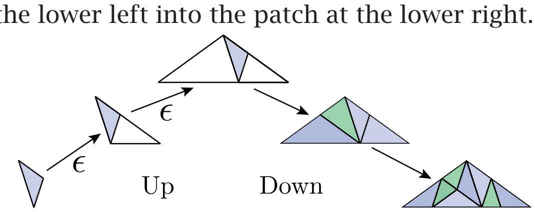
The inflated half-rhombs are uniquely composed of the original half-rhombs.



A directed graph, whose nodes correspond to half-rhombs, results. A directed edge records the containment of one half-rhomb in the inflated version of another.



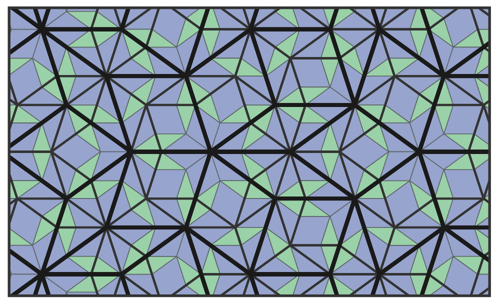
Through the process de Bruijn calls *updown generation*, an infinite path in the directed graph produces, with a few exceptions, a tiling. To illustrate, the figure below, which appears in [7], shows how the finite path $\epsilon \cdot \epsilon$ extends the half-rhomb at the lower left into the patch at the lower right.



Each new edge added to the path gives a way to continue playing Penrose's game.

Also, two paths define the same tiling precisely when they agree beyond some point, so the number of different Penrose tilings is uncountable.

Inflation, first described in Gardner's remarkable article [2], is crucial for understanding Penrose tilings. There is an infinite hierarchy of tilings, in which each tiling is the inflated tiling of its predecessor.



A translational symmetry of a tiling would also be a symmetry of the inflated tiling and hence of every tiling in the hierarchy. As the size of the tiles grows exponentially, however, this is not possible, and so Penrose tilings admit no translational symmetries. The inflation hierarchy, however, does create a sense of order that explains the diffraction patterns of so-called quasicrystals first created in laboratories a few years after Penrose's discovery (see [7]). Penrose tilings have also been used to make thicker, softer toilet paper [4].

References

Penrose's discovery is described in Gardner [2] and Penrose [5], and Grünbaum and Shephard [3] present the original work of Penrose, Conway, and Ammann. Much of what is known about rhomb tilings is due to de Bruijn.

[1] N. G. de Bruijn, Updown generation of Penrose tilings, *Indag. Math. (N. S.)* **1 (2)** (1990), 201–219.
[2] M. Gardner, Extraordinary nonperiodic tiling that enriches the theory of tiles, *Sci. Amer.* (January 1977), 110–121.
[3] B. Grünbaum and G. C. Shephard, *Tilings and Patterns*, W. H. Freeman, New York, 1987.
[4] S. Mirsky, The emperor's new toilet paper, *Sci. Amer.* (July 1997), 24.
[5] R. Penrose, Pentaplexity, *Eureka* **39** (1978), 16–22.
[6] ——— , Tilings and quasicrystals: a nonlocal growth problem?, in *Introduction to the Mathematics of Quasicrystals* (Marko Jarić, ed.), Academic Press, 1989, pp. 53–80.
[7] M. Senechal, *Quasicrystals and Geometry*, Cambridge University Press, Cambridge, UK, 1995.
[8] U. S. Patent and Trademark Office, `http://patft.uspto.gov/netahtml/srchnum.htm`, search for patent 4,133,152.
[9] `http://merganser.math.gvsu.edu/david/penrose`.