Book Review

Mathematics in Nature: Modeling Patterns in the Natural World

Reviewed by Brian D. Sleeman

Mathematics in Nature: Modeling Patterns in the Natural World
John A. Adam
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> And while I have sought to shew the naturalist how a few mathematical concepts and dynamical principles may help and guide him, I have tried to shew the mathematician a field for his labour—a field which few have entered and no man explored. Here may be found homely problems, such as often tax the highest skill of the mathematician, and reward his ingenuity all the more for their trivial associations and outward semblance of simplicity.

So writes D'Arcy Wentworth Thompson (1860–1948) [8], the great biologist, in the epilogue to his magnum opus *On Growth and Form* in 1917. In this classic work Thompson states his task in the following words:

> The terms Growth and Form, which make up the title of this book, are to be understood, as I need hardly say, in their relation to the study of organisms. We want to see how, in some cases at least, the forms of living things, and of the parts of living things, can be explained by physical considerations, and to realise that in general no organic form exists save as are in conformality with physical and mathematical laws.

The torch lit by D'Arcy Thompson has been taken up, in the last quarter of the twentieth century, by a growing band of mathematicians and theoreticians to the extent that mathematical or theoretical biology is well recognised as an important discipline in many undergraduate and graduate schools in universities and colleges. Mathematicians have indeed brought their skills to address biological questions. The Fields Medallist René Thom brought a great wealth of new topological and analytic ideas to the fundamental problems of modelling and understanding morphogenesis [7]. Here Thom introduces his idea of a catastrophe to build mathematical models of embryology, the structure of cells, as well as models of thought and language. In the two-volume work on mathematical biology [5], Murray has brought to bear a wealth of modelling ideas and mathematical techniques, ranging from the most elementary to the cutting edge of modern nonlinear mathematical analysis, to describe a vast array of biological phenomena. Indeed, Murray's books have had and continue to have a major impact on current mathematical biological research.

Much more recently, in this postgenomic era of biomedical research, a key objective is to systematically catalogue all the molecules and their interactions within a living cell. This in turn has given rise to the concepts of network biology, which finds its mathematical expression in terms of the theory of random graphs. Indeed, recent advances in network biology suggest that cellular networks are governed by universal laws and offer a

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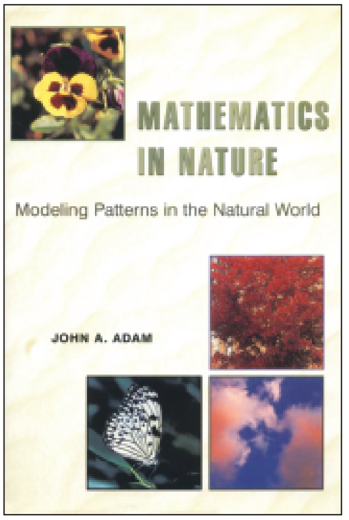
new conceptual framework that may revolutionise our view of biology [1], [2].

In the drive to bring these exciting developments to the lay audience and to aid the public understanding of science, eminent scientists have written some excellent books on the theoretical underpinning of biology. In this regard the books of Brian Goodwin, *How the Leopard Changed Its Spots: The Evolution of Complexity* [3]; Ian Stewart, *Life's Other Secret* [6]; and Hans Meinhardt, *The Algorithmic Beauty of Sea Shells* [4], are just a few examples.

At the start of the new millennium it is natural and even fortuitous that John Adam has written a book that in some ways attempts to light a new torch. However, the torch that Adam carries illuminates a different path from that of D'Arcy Thompson. John Adam's quest is a very simple one: that is, to invite one to look around and observe the wonders of nature, both natural and biological; to ponder them; and to try to explain them at various levels with, for the most part, quite elementary mathematical concepts and techniques. No mathematical technique, however sophisticated, can lead to a deeper understanding of the natural world unless the practitioner has been able to ask the right questions and to express the problem in terms of a mathematical model that can be explored and tested against experimental or observable evidence.

Mathematics in Nature begins by using simple arithmetical ideas to investigate some quite intriguing and fun problems. For instance, Adam relates the apparently true story of an inmate at a correctional center in West Virginia who escaped from the prison grounds by using a rope made from dental floss to pull himself over the courtyard wall. Given that the rope was estimated to be the thickness of a telephone cord (about 4 mm in diameter) and the wall 18 feet high, how many packets of dental floss were required? This question can be answered by estimating the diameter of dental floss to be 1/2 mm and noting that a typical packet contains 55 yards. A very different kind of problem is that of estimating the weight of the atmosphere, a calculation that might seem daunting but is in fact surprisingly simple. There is also the problem of saving the world from an alien attack! Here Adam recalls the sci-fi novel *The Black Cloud* by the late astronomer Sir Fred Hoyle, in which one of the characters does a "back-of-an-envelope" calculation to estimate the time of arrival of a mysterious and seemingly intelligent cloud of dust and gas that is directly approaching Earth.

Next, one is introduced to the ideas of dimensional analysis. This was the starting point for D'Arcy Thompson and essentially concerns the way in which physical characteristics vary with size. Dimensional analysis is perhaps the first modelling technique required in order to begin to understand any physical or natural phenomenon. Here we meet the problem of estimating how long a sea mammal can endure a dive or how high a flea can jump. Another interesting application is to use dimensional analysis to study the question, why do cells divide when they reach a certain size?



A wonder of nature that is familiar to everyone is the rainbow. It is one of the most beautiful and intangible manifestations of nature, and the history of its theorisation goes back to Descartes, Newton, and Snell. John Adam writes a very accessible account of the structure of rainbows using elementary trigonometry and basic differential calculus. In addition, he leads one into related phenomena such as the halo and the glory, the formation of ice crystals and snowflakes, and even a discussion of the iridescence of the wing cases of beetles. It is a delight to see how quite basic mathematical techniques can be used to help one understand and appreciate the beauty of the world around us.

In the chapter "Clouds, Sand Dunes, and Hurricanes" a variety of modelling ideas are introduced, and here Adam reemphasises his basic philosophy, i.e., "try to understand a given phenomenon at as many complementary levels as possible."

Waves occur in many situations in the natural world. Ocean waves and the ripples on a pond are commonly observed by everyone. The evidence for the presence of waves can sometimes be seen in cloud formations and the shape of sand dunes. Adam devotes considerable time to the study and structure of waves. Beginning with linear wave theory, he leads one to examine dispersion relations and matters of stability. There are applications of the theory to shallow and deep-water waves as well as to ship waves. There are of course limitations to the linear theory. An example of a wave phenomena that cannot be modelled by linear theory is that of a remarkable "solitary wave" observed by J. Scott-Russell in 1834:

> I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped—not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great

> velocity assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling at a rate of some eight to nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that singular and beautiful phenomenon.

This phenomenon led to a mathematical description in 1895 by Korteweg and de Vries and gave birth not only to the so-called KDV equation but also to the modern studies of integrable partial differential equations and the inverse scattering method.

Although KDV is beyond the scope of this book, there is a nice treatment of the somewhat more tractable Burger's equation, which models, for example, the mechanisms which maintain a tidal bore.

Once more one is invited to look around for more of nature's symmetries. Look up and you may see hexagonal convection-cell clouds; look around and you may see mud cracks of polygonal shape. Look at a bees' honeycomb with its very regular hexagonal cells. In this latter phenomenon John Adam argues that while bees are not optimisation experts, they do like to construct polygonal cells that enclose a region of maximum area but with minimum perimeter. This then leads to a discussion of the areas and perimeters of regular polygons and the problem of tiling two-dimensional space. It is easy to see that the hexagonal cell does the trick here. However, when one recognises that honeycomb cells are essentially three-dimensional, the problem is much more challenging and involves the so-called Kelvin Problem: What is the most efficient (i.e., minimum of the surface area of the boundary) partition of space into equal volumes? It is surprising how far one can investigate such problems with the use of elementary geometry and calculus. In the same chapter Adam takes up a similar line of enquiry in the discussion of soap bubbles and foams, which is the term used for an agglomeration of bubbles. Polygonal mud cracks are a different phenomenon altogether, and different modelling ideas are required. A good start is to recognise that tension forces are involved, and this suggests that Hooke's law—namely, tension is proportional to displacement—is a good idea, combined with the principle of least work.

In Chapter 12 Adam looks at the problems of meandering rivers and river draining patterns, as well as the mechanical forces experienced by trees. In the former case an appeal is made to the hypothesis due to Langbein and Leopold that "meanders are not mere accidents of nature, but the form in which a river does the least work in turning." Here Adam considers the mean square curvature of a river bend and uses the basic ideas drawn from the calculus of variations to obtain conditions under which the mean square derivative of the curvature is a minimum. He also draws out a connection with stresses in an elastic wire.

In the discussion of river drainage patterns one is introduced to the cellular automata rules which are invoked. A river or stream is said to be of class 1. When any two rivers merge they form a branch of class 2, and after merging with yet another river, they form a branch of class 3, and so on. This can be generalised through defining the following rules:

1. If two tributaries of the same class i merge, the resulting branch is of class $i+1$.

2. If two tributaries of different classes, i and j, merge, where $j > i$, the resulting branch is of class j.

The next step is to consider N_i, the total number of tributaries of class i, and let m be the class of the main stream. Then by assuming that N_i obeys simple power laws or, more interestingly, is equal to a particular Fibonacci number, then some quite realistic drainage patterns may be computed. Adam then goes on to use beam theory to discuss the bending and shaking of trees as well as to estimate how high a tree can grow without buckling under its own weight.

Bird flight is the subject of Chapter 13. Here it is interesting to learn that it is much more efficient for flocks of birds to fly in V formations rather than individually. Adam develops the basic equations of bird flight, which involves the concepts of drag, lift, and wingloading. Using these concepts together with dimensional analysis and Bernoulli's theorem, he discusses gliding and hovering, and also describes how soaring birds take advantage of thermals and sea birds of wind shear just above the surface of the sea.

How did the leopard get its spots? One answer to this question is given by Rudyard Kipling in his *Just So* stories. While this account is a delightful legend, it does not help us to understand how animal coat markings arise in general. For this we turn to the great scientist Alan Turing of *Enigma* fame and, to some, better known as a founding father of computing. In a groundbreaking paper [9] that appeared in 1952, he showed that diffusion can destabilise a chemical concentration to produce patterns in place of a uniform homogeneous steady state. This rather counterintuitive observation has led to an enormous body of work on the study of reaction-diffusion equations with far-reaching biological consequences. J. D. Murray [5] and many others have taken up Turing's ideas and developed many models of animal coat markings, limb-bud development, and even how the crocodile got its teeth!

The underlying assumptions of patterning models are:

1. Certain chemicals (called morphogens) stimulate cells to produce melanin, high concentrations of which produce colouration, but low concentrations do not.

2. Two chemicals are produced in the skin. One activates the production of melanin, and the other inhibits it.

3. Production of the "activator" initiates the production of the "inhibitor".

4. The inhibitor diffuses faster than the activator.

These assumptions, combined with Turing's discovery, lead to the construction of activator-inhibitor systems of reaction-diffusion equations that can be used to model a wide range of patternings in nature.

As Adam points out, there are alternatives to the reaction-diffusion embryological pattern-forming models. One example, not discussed in this book, is Lewis Wolpert's [10] idea of "positional information", suggesting that cells are preprogrammed to read a chemical (i.e., morphogen) concentration and differentiate accordingly into different kinds of cells destined to become, for example, cartilage, bone, tissue, hair, etc. Indeed, there is still much controversy surrounding this fundamental problem of developmental biology.

A related and fascinating study modelled by reaction-diffusion equations is the colouring and patterning of a butterfly wing. The wing pattern is laid down during the pupation stage. One suggestion is that a morphogen that "switches on" a particular gene in the wing cells is released from sources located somewhere on the wing. The morphogens diffuse throughout the wing cells and "throw" biochemical switches when they exceed some threshold concentration. It also turns out that the wing pattern depends crucially on the geometry and scale of the wing.

Adam illustrates these ideas in application to a simple one-dimensional model of diffusing morphogen that introduces the fundamental solution of the heat equation and the method of separation of variables. There is also a brief discussion of the development of plankton blooms.

The book concludes with a little appetising dip into fractal geometry, which is currently reshaping some of the ways one thinks about patterns in nature.

Unlike D'Arcy Thompson, John Adam did not write an epilogue to his book, but it does deserve one. To paraphrase D'Arcy Thompson, such an epilogue could read:

> *And while I have sought to show the natural observer how a few mathematical concepts and dynamical principles may help and guide him, I have tried to show students and practitioners of mathematics and the just plain curious a field of adventure for their labour. Here may be found homely problems to tax the highest skills of mathematical students and reward their ingenuity.*

On Growth and Form is a classic; *Mathematics in Nature* has the potential to become one too.

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