# The $p$-adic Icosahedron

*Gunther Cornelissen and Fumiharu Kato*

**Editor's Note:** The *Notices* has been pleased to feature color graphics. With this article, we initiate color text.

—Andy Magid

The last proposition in Euclid's *Elements* states that there are only five convex regular polyhedra, called the *platonic solids* (after Platon, who listed them in his Timaios around 350 B.C.): tetrahedron, cube, octahedron, icosahedron, and dodecahedron. This very short list of extremely beautiful and regular mathematical objects has mystified scientists for many centuries. For instance, Kepler's *Mysterium Cosmographicum* from 1597 (wrongly) models distances between planets in the solar system using the platonic solids (Figure 1). More recently, Grothendieck is supposed to have said, "The platonic solids are so beautiful and exceptional that one cannot assume such exceptional beauty will hold in more general situations." (*Notices*, vol. 51, no. 10, p. 1196). The platonic solids fit into the larger picture of "ADE-classification(s)" and the theory of finite reflection groups, as is very well explained in John Baez's Week 62 Finds [2].

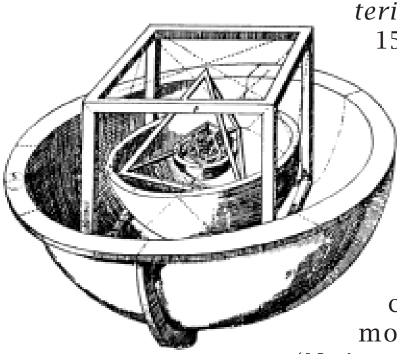


**Figure 1. The solar system according to Kepler's *Mysterium Cosmographicum.***

The solids belong to the "real" world: they are part of geometry over the real numbers. But, following Hensel and Ostrowski, it has been known for more than seventy years that the real numbers form only one of the many possible completions of the field of rational numbers, the other possibilities being given by the $p$-adic numbers. So it seems only natural to ask for the analogue of the platonic solids in the $p$-adic world. The question turns out to be relevant for the study of $p$-adic "orbifolds" and number-theoretic properties of, for instance, solutions to hypergeometric differential equations. We will illustrate the construction using our own favorite polyhedron, the icosahedron, the only platonic solid (with its dual polyhedron, the dodecahedron) with a simple group of symmetries. After that we will show some examples from the theory of $p$-adic uniformization.

*Gunther Cornelissen is lecturer/researcher at the University of Utrecht, The Netherlands. His email address is* cornelis@math.uu.nl.

*Fumiharu Kato is associate professor of mathematics at Kyoto University, Japan. His email address is* kato@math.kyoto-u.ac.jp.

*The authors have written some ten papers on nonarchimedean uniformization. This text is based on a lecture of the second author at the Mathematical Society of Japan meeting of September 2003 held at Chiba University.*

## The Real Icosahedron

We will first recall a construction of the usual icosahedron. Let us agree that when discussing a figure in the text, we relate part of the picture to part of the text by giving both the same color (red, green, or blue—up to Figure 11). The icosahedron in Figure 2 consists of 12 vertices, 30 edges with 30 midpoints, and 20 faces (each of which is an equilateral triangle) with 20 barycenters.

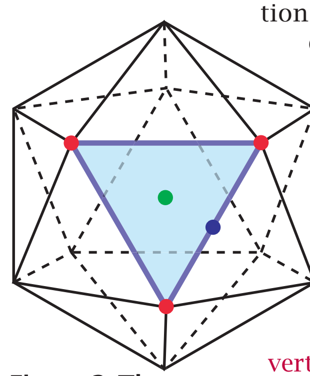


**Figure 2. The icosahedron.**

Projecting the icosahedron from its center to a circumscribed sphere maps each edge onto a part of a geodesic line on the sphere. These geodesic lines intersect at vertices, midpoints of edges, or barycenters of faces. They provide a tessellation of

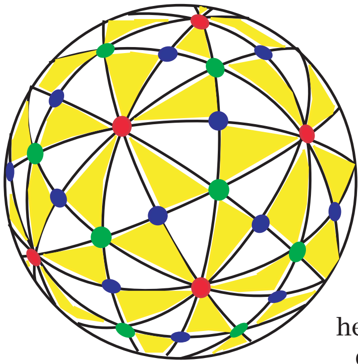


Figure 3. Icosahedral tessellation of the Riemann sphere.

the sphere by 120 triangles with angles ($\pi/2$, $\pi/3$, $\pi/5$) (Figure 3). Observe that one would arrive at the same tessellation starting from the dodecahedron (which is the dual polyhedron to the icosahedron), but with green vertices: in Figure 3, connecting five adjacent green spots reveals a pentagon, the face of a dodecahedron.

Let $A_5$ denote the group of even permutations on five letters, i.e., the simple group with 60 elements. An *icosahedral group* is a copy of $A_5$ embedded in $PGL(2, \mathbf{C})$, the automorphism group of the Riemann sphere (= complex fractional linear transformations). All icosahedral groups turn out to be conjugate, and one of them is given explicitly by generators as follows:

$$I = \left\langle \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} \zeta + \zeta^{-1} & 1 \\ 1 & -(\zeta + \zeta^{-1}) \end{pmatrix} \right\rangle,$$

where $\zeta$ is a primitive fifth root of unity (e.g., $\zeta = e^{2\pi i/5}$). The icosahedral group is the orientation-preserving symmetry group of the icosahedron; it has 6 cyclic subgroups of order 5, 10 cyclic subgroups of order 3, and 15 cyclic subgroups of order 2. The respective fixed points of these subgroups on the Riemann sphere are the 12 vertices, 20 barycenters of faces, and 30 midpoints of edges of the icosahedron.

The quotient of the Riemann sphere by the group $I$ is again a Riemann sphere. The quotient map is branched above three points on the sphere, with branching degrees 2, 3, and 5 respectively (Figure 4). The ramification points over the branch points of degrees 5, 3, and 2 are exactly the vertices, barycenters of faces, and midpoints of edges. Each pair consisting of a white and yellow triangle is a fundamental domain for the action of $I$, and each triangle is mapped onto one of the half-planes in the quotient.

The multivalued function inverse to this covering map can be written as the ratio of two independent solutions to the Gaussian hypergeometric equation $E(a, b, c)$:

$$x(1-x)\frac{d^2u}{dx^2} + [c - (a+b+1)x]\frac{du}{dx} - abu = 0,$$

where the constants $a, b$, and $c$ are any rational numbers such that

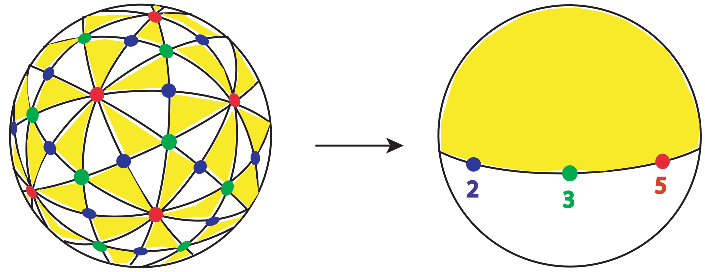



Figure 4. Quotient of the Riemann sphere by $I$.

$$\{|1-c|, |c-a-b|, |a-b|\} = \left\{\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{5}\right\}$$

(e.g., $a = 11/60$, $b = -1/60$, $c = 1/2$).

## $p$-adic Numbers: A Brief Recap

One construction of the real numbers is by considering them as the completion of the rational numbers $\mathbf{Q}$ w.r.t. the usual absolute value, i.e., by adding to $\mathbf{Q}$ all (limits of) Cauchy sequences modulo null-sequences. A famous theorem of Ostrowski states that, up to a certain natural equivalence, $\mathbf{Q}$ carries exactly the following further absolute values: let $p$ be a prime number; for $q \in \mathbf{Q}$, define $|q|_p = p^{-e}$ if we can write $q = p^e \cdot u$ with $u$ a rational number without $p$ in the numerator and the denominator. In this context a number is small precisely when it is highly divisible by $p$. The completion of $\mathbf{Q}$ w.r.t. $|\cdot|_p$ is called the set of *p-adic numbers* and carries a natural operation of addition and multiplication that make it into a complete field. As a typical example, the sum $1 + p + p^2 + p^3 + \cdots$ (that diverges in $\mathbf{R}$) converges in the $p$-adic numbers, since for its general term, $|p^i|_p = p^{-i} \to 0$ as $i \to +\infty$. The sum actually equals the rational number $1/(1-p)$, but of course there are many more $p$-adic numbers than just rational ones. A general $p$-adic number $q$ can be expanded as a "convergent Laurent series in $p$": $q = a_N p^N + a_{N+1} p^{N+1} + \ldots$ for some integer $N$ (possibly negative) with $a_i \in \{0, 1, \ldots, p-1\}$. The set $\mathbf{Z}_p$ of $p$-adic integers consists of those $p$-adic numbers $q$ for which $|q|_p \leq 1$. These are exactly the numbers for which only positive powers of $p$ occur in the above "Laurent series" (so $N$ can be chosen a positive integer).

## The $p$-adic Riemann Sphere

In order to find the $p$-adic analogue of the icosahedron, we just have to look at the construction above: one finds the fixed points of an action of the cyclic subgroups of $A_5$ on the Riemann sphere. So first of all, we have to introduce the $p$-adic analogue of the Riemann sphere, which is an analytic structure on the projective line $\mathbf{P}^1$. The naive way of "doing analysis with the $p$-adic metric on the coordinates" does not work (because of total disconnectedness). One of the most natural ways of putting a genuine $p$-adic analytic structure on $\mathbf{P}^1$ is by regarding it as a Berkovich space, but as the phenomena we are interested in are already visible at the level of the "skeleton" of that space, we will content ourselves with a description at the level of trees [9]. The *Bruhat-Tits tree* $\mathcal{T}$ of $PGL(2, \mathbf{Q}_p)$ is a graph, technically defined as follows:

- Vertices are $\mathbf{Q}_p^*$-homothety classes $[M]$ of $\mathbf{Z}_p$-lattices $M$ in $\mathbf{Q}_p^2$.
- Two vertices $[M]$ and $[N]$ are joined by an edge if and only if representatives $M$ and $N$ can be chosen such that $pM \subset N \subset M$.

The graph $\mathcal{T}$ is actually a tree, and edges emanating from any given vertex are in one-to-one correspondence with $\mathbf{F}_p$-rational points of $\mathbf{P}^1$.

The graph $\mathcal{T}$ is a regular $(p+1)$-valent tree, as in Figure 5 (where $p = 2$ for simplicity).

We will actually need a slight extension of this definition, because we will want the fixed points of elements of order 2, 3, and 5 from $A_5$ acting on $\mathbf{P}^1$ to be defined over the field we are working with, and $\mathbf{Q}_p$ itself is not always good enough for that. So we let $K$ be a finite field extension of $\mathbf{Q}_p$ that contains a primitive third, fourth, and fifth root of unity (i.e., all roots of $X^{60} - 1$). The $p$-adic absolute value extends uniquely to $K$; denote this extension by $|\cdot|$. Let $\mathcal{O}_K$ be the set of integers in $K$: $\mathcal{O}_K := \{x \in K : |x| \le 1\}$, which actually turns out to be a ring. Let $\pi$ denote a uniformizer of $K$, i.e., a generator of the ideal $\{|x| < 1\}$ in $\mathcal{O}_K$. Let $k$ denote the residue field of $K$: $k = \mathcal{O}_K/\pi$; it is actually a finite field. All the above definitions make sense if one replaces $\mathbf{Q}_p$ by $K$, $\mathbf{Z}_p$ by $\mathcal{O}_K$, and $p$ by $\pi$. One arrives at the Bruhat-Tits tree $\mathcal{T}$ of $PGL(2, K)$.

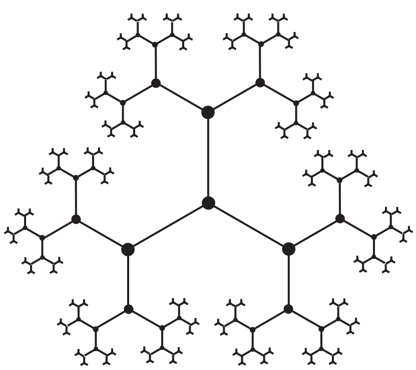


**Figure 5. Part of the Bruhat-Tits tree of $PGL(2, \mathbf{Q}_2)$.**

The *ends* of $\mathcal{T}$ are equivalence classes of infinite half lines in $\mathcal{T}$ (two of which are identified if they only differ in finitely many edges), and these ends are in one-to-one correspondence with $K$-rational points of $\mathbf{P}^1$. Since $PGL(2, K)$ acts on homothety classes of lattices in $K^2$, from its definition the tree $\mathcal{T}$ acquires a natural action of the group $PGL(2, K)$, and via this correspondence its action on the ends of $\mathcal{T}$ is the same as its natural action on $\mathbf{P}^1(K)$.

We then have a way to connect points of the (topologically totally disconnected) space $\mathbf{P}^1(K)$ (seen as ends of $\mathcal{T}$) via infinite paths inside $\mathcal{T}$, and these paths play the rôle of the geodesics in the original "real" picture. To visualize this and the correspondence between ends of $\mathcal{T}$ and points of $\mathbf{P}^1(K)$ in a clearer way, we will now draw $\mathbf{P}^1(K)$ as an actual compact "sphere" and put the tree $\mathcal{T}$ inside it; see Figure 6.

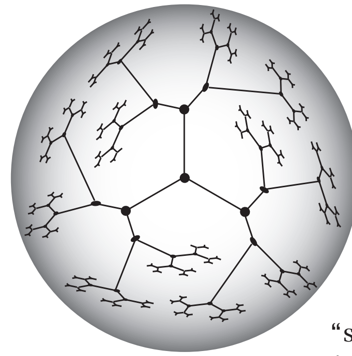


**Figure 6. The Bruhat-Tits tree of $PGL(2, \mathbf{Q}_2)$ inside $\mathbf{P}^1(\mathbf{Q}_2)$.**

For any compact (e.g., finite) subset $\Sigma$ of $\mathbf{P}^1(K)$, we define a subtree $\mathcal{T}_\Sigma$ of $\mathcal{T}$ that is minimal amongst all subtrees of $\mathcal{T}$ having $\Sigma$ as its set of ends. For example, if $\Sigma$ consists of three points, the subtree $\mathcal{T}_\Sigma$ is a "tripod", as in Figure 7.

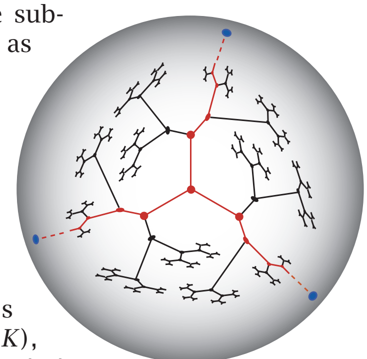


**Figure 7. A tripod.**

## The $p$-adic Icosahedron

The matrix representation of the group $I$ from the first section makes sense in $PGL(2, K)$, since $K$ contains a primitive fifth root of unity $\zeta$. Let $\Sigma$ denote the set of points of $\mathbf{P}^1(K)$ fixed by a nontrivial element of $I$. There are 12 points fixed by at least one of the elements of order 5 in $I$, 20 points fixed by at least one element of order 3 in $I$, and 30 points fixed by at least one element of order 2 in $I$; and these points together form all of $\Sigma$. We call the subtree associated to this $\Sigma$ the *$p$-adic icosahedron*: it is the structure that arises fromconnecting the fixed points of elements of $I$ via geodesics in the $p$-adic Riemann sphere, so its construction really parallels that of the real icosahedron. The $p$-adic icosahedron is a tree with $12 + 20 + 30 = 62$ ends. Since the actual "position" of points on $\mathbf{P}^1(K)$ does not make sense, it is the *combinatorial structure of $\mathcal{T}_\Sigma$, together with a labeling of vertices and edges by their stabilizers*, that is of interest. We will do this labeling as follows: if a group is cyclic, we just label by its order; otherwise, we indicate the full group. Ends are indicated by edges carrying arrows, and the stabilizer of the end is written as a label on the arrow. Finally, if two vertices are stable by the same group, it is understood that the same holds for the connecting edge.

If $p > 5$, the $p$-adic icosahedron is rather boring: it is the union of half-lines corresponding to the points of $\Sigma$ emanating from a single vertex, the unique vertex fixed by the whole of $I$, which we call the *center* of $\mathcal{T}_\Sigma$; see Figure 8.

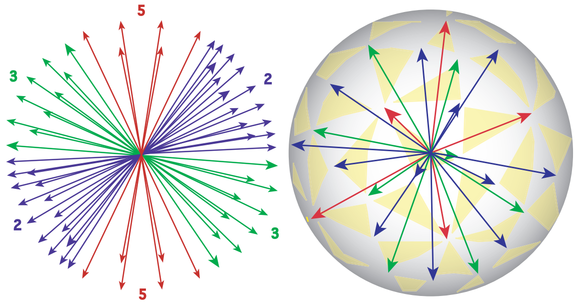



**Figure 8. The $p > 5$-icosahedron.**

In the next three pictures we consider what happens if $p \le 5$. The icosahedron is the right part of each figure, and the left part is a zoom-in on the "horizontal" part of the right picture.

If $p = 5$, the six lines fixed by the six cyclic subgroups of order 5 (so-called *mirrors* of order 5) are separated from the center by fixed points of a copy of $D_5$ in $I$; see Figure 9.

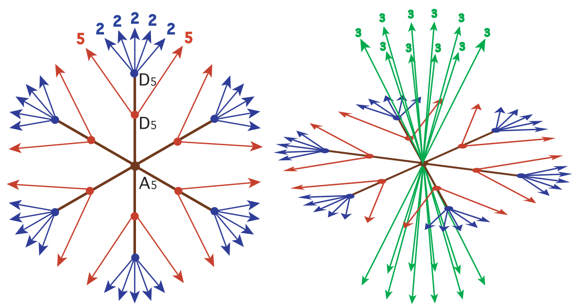



**Figure 9. The pentadic icosahedron.**

If $p = 3$, 10 mirrors of order 3 are separated from the center, as in Figure 10.

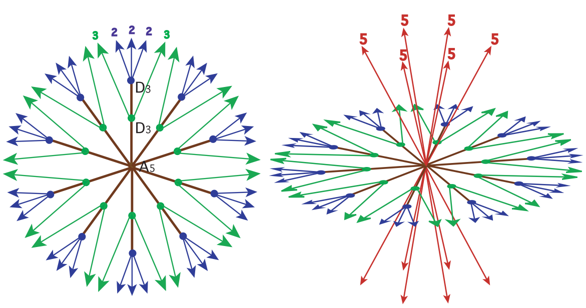



**Figure 10. The triadic icosahedron.**

Finally, if $p = 2$, the 15 mirrors of order 2 are separated from the center and from the mirrors of order 3; see Figure 11. Kazuya Kato from Kyoto University proclaimed that these objects should be seen as "the flowers in his $p$-adic garden". The

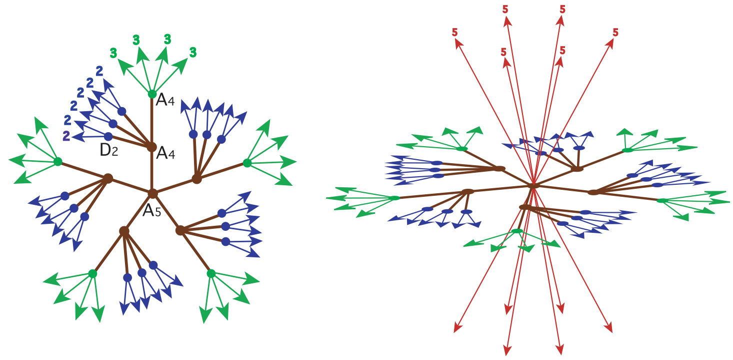



**Figure 11. The dyadic icosahedron.**

special status of the primes $p \le 5$ will appear over and over again in the sequel.

## Application: $p$-adic Orbifolds

One can perform the previous construction starting from the other finite subgroups of $PGL(2, K)$ (such as $A_4, S_4$, etc.) to arrive at the full list of $p$-adic polyhedra. This is because the full list of finite subgroups of $PGL(2)$ over any field is known up to conjugation: over a field of characteristic zero such as $K$, the list is identical to that over the complex numbers; over a field of positive characteristic (which we will briefly touch upon later), there is Dickson's famous classification. In the end, all $p$-adic (or nonarchimedean) polyhedra can thus be explicitly drawn by the techniques from the previous section. They play a rôle in the construction of $p$-adic orbifolds and in classification results related to those.

Let us first briefly recall Mumford's theory of nonarchimedean uniformization of curves (cf. [5]), which is a mixture of the usual uniformization theory of Riemann surfaces by embedding the fundamental group in $PSL(2, \mathbf{R})$ and the theory of Schottky uniformization. The general setup is a bit technical. Let $K$ be a general nonarchimedean valued field, complete w.r.t. an absolute value that satisfies the strong triangle inequality $|a+b| \le \max\{|a|, |b|\}$ for all $a, b \in K$; for example, $(K, |\cdot|) = (\mathbf{Q}_p, |\cdot|_p)$. If $\Gamma$ is a torsion-free finitely generated discrete subgroup of $PGL(2, K)$, let $\mathcal{L}_\Gamma$ denote the set of limit points for the action of $\Gamma$ on $\mathbf{P}^1(K)$, let $\Omega_\Gamma = \mathbf{P}^{1,\mathrm{an}}_K - \mathcal{L}_\Gamma$, and let $\mathcal{T}_\Gamma = \mathcal{T}_{\mathcal{L}_\Gamma}$; here the superscript "an" refers to a $K$-analytic structure on $\mathbf{P}^1$. Then $X_\Gamma = \Omega_\Gamma / \Gamma$ can be given the structure of a smooth projective curve over $K$. Technically speaking, this curve admits a semistable model over the integers of $K$ whose special fiber is a union of $k$-rational curves intersecting in $k$-rational points. The intersection dual graph of this special fiber is isomorphic to $\mathcal{T}_\Gamma$. Conversely, every curve over $K$ with split multiplicative reduction is isomorphic to a curve obtained via such a construction; such a

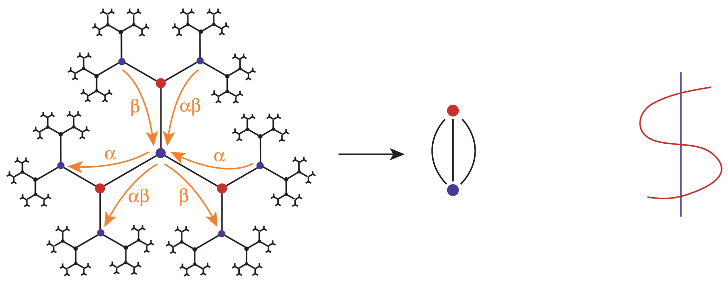



**Figure 12. Example of $\mathcal{T}_\Gamma$, $\mathcal{T}_\Gamma/\Gamma$, and the reduction of $X_\Gamma$.**

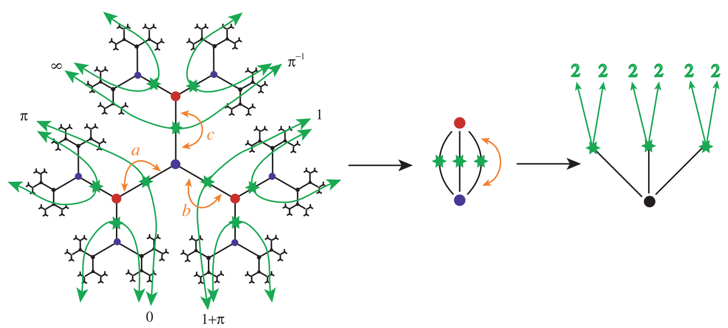



**Figure 13. Action of $N$ on $\mathcal{T}_N^*$, $\mathcal{T}_N/\Gamma$, and $\mathcal{T}_N^*/N$.**

curve is called a *Mumford curve*. Let us clarify these concepts on an example; see Figure 12.

Assume $K$ does not have characteristic two, and fix $\pi$ with $0 < |\pi| < 1$. Define a subgroup $\Gamma$ of $PGL(2,K)$ generated by two elements

$$\alpha = \begin{pmatrix} \pi(2+\pi) & -2\pi(1+\pi) \\ 4 & \pi^2 - 2\pi - 4 \end{pmatrix},$$

$$\beta = \begin{pmatrix} -\pi(2+\pi) & 2(2+\pi^2) \\ -2\pi & -\pi^2 - 2\pi + 4 \end{pmatrix}.$$

Part of the associated tree $\mathcal{T}_\Gamma$ with the action of some elements from $\Gamma$ is shown in Figure 12, together with the quotient $\mathcal{T}_\Gamma/\Gamma$ and the reduction of the corresponding curve $X_\Gamma$ (which is intersection dual to the quotient graph in the following sense: lines in the reduction are replaced by vertices in the graph, and vertices are connected if and only if the corresponding lines intersect). This group $\Gamma$ is free of rank *two*, the corresponding curve has genus *two*, and its reduction is a "dollar" sign (which has *two* holes, just like $\mathcal{T}_\Gamma/\Gamma$). The fact that the number two occurs three times in the previous sentence is no coincidence! The algebraic curve $X_\Gamma$ has some equation

$$y^2 = (x - a_1)\dots(x - a_6),$$

and a suitable change of coordinates $(x, y)$ leads to a good model, of which the special fiber is gotten by reducing everything modulo $\pi$.

More generally, if $N$ is any (not necessarily free) finitely generated discrete subgroup of $PGL(2,K)$ with set of limit points $\mathcal{L}_N$, let $\mathcal{F}_N$ denote the set of all *fixed* points of elements of finite order in $N$. Let $\mathcal{T}_N^* = \mathcal{T}_{\mathcal{L}_N \cup \mathcal{F}_N}$.

- If $N$ is torsion-free, then $\mathcal{T}_N^* = \mathcal{T}_N$.
- If $N$ is finite, then $\mathcal{L}_N = \emptyset$, so $\mathcal{T}_N = \emptyset$, but $\mathcal{T}_N^*$ is a $p$-adic polyhedron.

If we let $\Omega_N = \mathbf{P}_K^{1,\mathrm{an}} - \mathcal{L}_N$, then the quotient $X_N := \Omega_N/N$ carries a structure of smooth projective curve. Actually, there exists a finite index normal free subgroup $\Gamma \subset N$, and $X_N$ is the quotient of the Mumford curve $X_\Gamma$ by the finite group $N/\Gamma$.

In our example, $X_\Gamma$ is a curve of genus two that admits an automorphism of order two (on the above equation, it is just $y \mapsto -y$), and this can be seen within the above framework of "orbifold uniformization" as follows: we let $N$ be generated by three elements:

$$a = \begin{pmatrix} \pi & 0 \\ 2 & -\pi \end{pmatrix}, \; b = \begin{pmatrix} 2+\pi & -2(1+\pi) \\ 2 & -2-\pi \end{pmatrix},$$

$$c = \begin{pmatrix} \pi & 2 \\ 0 & \pi \end{pmatrix}.$$

With $\alpha = ab$ and $\beta = bc$, $N$ is isomorphic to the free product of three copies of a cyclic group of two elements having $\Gamma$ as normal subgroup of index two. We have displayed the corresponding pictures in Figure 13. Now $X_\Gamma$ is a double cover of $X_N \cong \mathbf{P}^1$ branched above six points (the ends of $\mathcal{T}_N^*/N$).

## Application: Classification Results

In the correspondence between $N$ and $\mathcal{T}_N^*$ (of which an example was given above), the ends of $\mathcal{T}_N^*/N$ correspond to the points of $X_N$ over which the map $X_\Gamma \to X_N$ is branched, and the stabilizer of such an end is exactly the ramification group of the corresponding point in the cover. This is a general phenomenon that can be used in the opposite direction: suppose one does not start from a group $N$, but one instead is interested in classifying coverings of the projective line $\mathbf{P}^1$ over $K$ that are branched above a fixed number of points with given ramification groups and such that the cover is a Mumford curve. Example: coverings of $\mathbf{P}^1$ that are branched over four points with ramification indices $(2,2,2,3)$. The strategy to classify them is then as follows (see Figure 14):

- One constructs the list of $p$-adic polyhedra (="atoms") $\mathcal{T}_N^*$ and draws the graphs $\mathcal{T}_N^*/N$.
- One tries to bond together these graphs $\mathcal{T}_N^*/N$ along common ends by folding them together (="chemical compound"). In this procedure, one assures that in the bond only four ends remain (with the correct stabilizers). On the level of

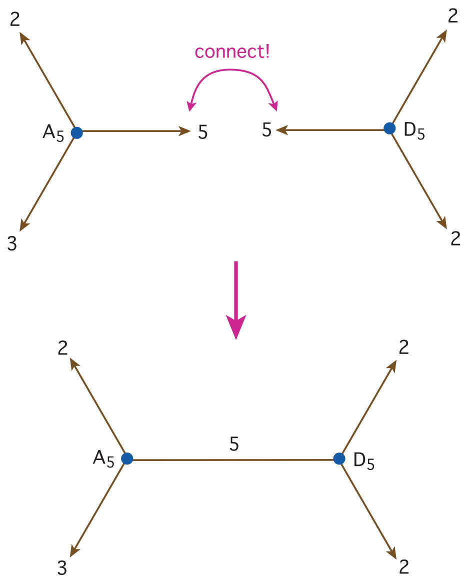



**Figure 14. "Inorganic chemistry method".**

abstract groups, this process is known as "amalgamation" of finite groups (cf. [9]).

- One makes sure that the corresponding group exists as a discrete subgroup of $PGL(2, K)$ (="existence problem of the compound"; cf. [8]).

In the example of $(2, 2, 2, 3)$-branched coverings of $\mathbf{P}^1$ for $p > 5$, one gets a list of possible abstract structures of the group $N$, as in Figure 15. Each of these can be realized by a Mumford curve. As a matter of fact, (A), (C), and (D) are realized by $p$-adically open loci in a pencil of genus six curves on a Del Pezzo quintic surface studied by Edge in the early 1980s [7].

It follows from the Riemann-Hurwitz-Zeuthen formula that curves $X$ with many symmetries (large automorphism group) arise as coverings of $\mathbf{P}^1$ branched above at most three points. If $X$ is a Mumford curve of genus $g$, then our classification can be used to bound the number of such symmetries, since for $X = X_\Gamma$, $\mathrm{Aut}(X) = N/\Gamma$ with $N$ the normalizer of $\Gamma$ in $PGL(2, K)$. We can recover the following result of Herrlich: If $X$ is a Mumford curve of genus $g \geq 2$ over a $p$-adic field, then the following sharp bound holds: $\mathrm{Aut}(X) \leq c \cdot (g - 1)$ with $c = 12$ if $p > 5$, $c = 30$ if $p = 5$, $c = 24$ if $p = 3$, and $c = 48$ if $p = 2$ (again, $p \leq 5$ is exceptional).

There is no reason to restrict to $p$-adic fields $K$; one might as well develop the theory if $K$ is a non-archimedean valued field of positive characteristic [4], such as $K = \mathbf{F}_q((t))$. Then $PGL(2, K)$ has more finite subgroups, but all polyhedra can be classified. There is, for example, such an exotic specimen as the $N = PGL(2, 7)$-polyhedron in characteristic 7, whose $\mathcal{T}_N^*/N$-graph has two ends: one fixed by the subgroup of upper triangular

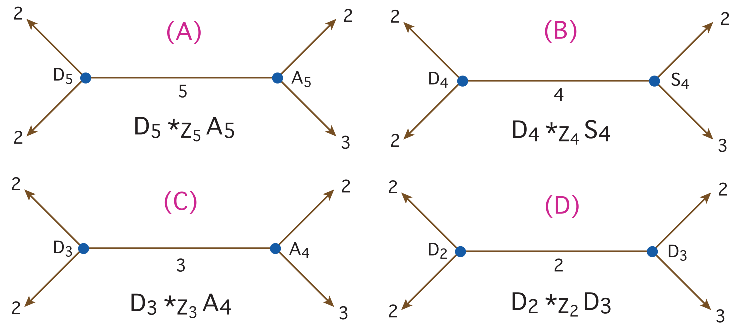



**Figure 15. Four possible $(2, 2, 2, 3)$-coverings.**

matrices, and one fixed by a cyclic subgroup of order 8.

Again, this list of polyhedra can be used to classify coverings of $\mathbf{P}^1$ by Mumford curves branched above two or three points in positive characteristic and leads to a sharp upper bound for the number of automorphisms of such Mumford curves $X$: *if $X$ is a Mumford curve of genus $g$ over a field of positive characteristic, then*

$$\mathrm{Aut}(X) \leq \max\{12(g-1), 2\sqrt{g}(\sqrt{g}+1)^2\}.$$

Returning to the case where $K$ is $p$-adic, if $N$ produces a covering of $\mathbf{P}^1$ branched above exactly three points, one calls $N$ a *$p$-adic triangle group (of Mumford type)*. The first ones were found by Yves André [1]. By the inorganic chemistry method above, the second named author has shown that they exist only if $p = 2, 3$, or $5$ (again, $p \leq 5$) and that there are infinitely many such. In Figure 16 we display the tessellation of the unit disk corresponding to a classical $(2, 4, 6)$-triangle group together with its triadic companion. The parallel is as follows: the classical group has limit points on the boundary of the Poincaré unit disk that correspond to the brown ends in the triadic case. The fixed points of elements of finite order (2, 4, or 6) in the disk are the blue, green, and red vertices of the triangles occurring in the tessellation (we have colored them in only one triangle), whereas in the triadic case the fixed points of elements of finite order (2, 4, or 6) are the blue, green, and red ends.

The classical Riemann-Hilbert correspondence between representations of the orbifold fundamental group and differential equations has a $p$-adic analogue via the theory of tempered coverings due to Yves André (cf. [1]). Our particular triangle group $\Delta(2, 4, 6)$ corresponds in this way to the Gaussian hypergeometric differential equation $E(\frac{1}{24}, \frac{7}{24}, \frac{5}{6})$:

$$x(1-x)\frac{d^2u}{dx^2} + \left(\frac{5}{6} - \frac{4}{3}x\right)\frac{du}{dx} - \frac{7}{576}u = 0.$$

The fact that this differential equation arises as above implies that the ratio of two nonproportional 3-adic solutions to the equation can be globally continued on a finite cover of $\mathbf{P}^1$: this is a very rare phenomenon in the $p$-adic situation, where there is no good analogue of analytic continuation.

## The "Conformal-Hyperbolic" Dictionary

Various invariants of the covering $X_\Gamma \to X_N$ of algebraic ("conformal") curves can be computed group theoretically ("hyperbolically") on $\mathcal{T}_N^*$. For example, the genus $g$ of the curve $X_N$ is the same as the cyclomatic number $c$ of the quotient graph $\mathcal{T}_N^*/N$ (the smallest number of edges which must be removed such that no circuit remains):

$$g(X_N) = c(\mathcal{T}_N^*/N).$$

As a second example, we have the following combinatorial group theory formula of Karass, Pietrowski, and Solitar [6]:

$$\frac{g(X_N)-1}{|N/\Gamma|} = \sum_{e\in E}\frac{1}{|N_e|} - \sum_{v\in V}\frac{1}{|N_v|},$$

where $E$ is the set of edges and $V$ is the set of vertices of $\mathcal{T}_N/N$ (and $N_*$ is the stabilizer of $*$ for the action of $N$ on some lift of $*$ to $\mathcal{T}_N$). The left-hand side of this formula is "algebro-geometric", since $N/\Gamma$ is the (finite) covering group of $X_\Gamma \to X_N$.

Another instance occurs when one computes in two ways the equivariant deformation space of a finite group acting on a Mumford curve. As an example, an embedding of a group of abstract type $N = (A)$ as in Figure 15 is given by conjugating the $A_5$ to its standard form "$I$"; any dihedral group of order ten that shares the cyclic group of order 5 generated by $\left(\begin{smallmatrix}\zeta & 0\\ 0 & 1\end{smallmatrix}\right)$ with this $I$ is then embedded in $PGL(2,K)$ by

$$D_5 = \left\langle \begin{pmatrix}\zeta & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & t\\ 1 & 0\end{pmatrix}\right\rangle,$$

where $t$ is a free "deformation parameter" restricted only by the condition that $D_5$ should be embedded discretely; for varying $t$ this produces a nonisotrivial family of Mumford curves of genus six with $A_5$ as automorphism group. Now on the algebraic side, we know that such a curve is a cover of $\mathbf{P}^1$ branched over four points with ramification indices $2, 2, 2, 3$. Fixing three of these points by an automorphism of $\mathbf{P}^1$, we are left with one degree of freedom: the location of the fourth point. Again, we see that the deformation space is one-dimensional. In [3] one finds two calculations of the dimension of this deformation space (even in positive characteristic): one in the "conformal" world of algebraic curves and one in the "hyperbolic" world of the graphs $\mathcal{T}_N$. In characteristic zero (where the formula has also been found by Herrlich), the formula says that the number of points $b$ of $X_N$ above which there is ramification equals

$$b(X_N) = \sum_{v\in V} d(N_v) - \sum_{e\in E} d(N_e),$$

where $d(G) = 2$ if $G$ is cyclic and $d(G) = 3$ otherwise. The fact that these two numbers agree in general appears as a statement in combinatorial group theory of which we do not know a direct proof (in the example, we get $4 = 3 + 3 - 2$). It is interesting to note that the computation of the equivariant deformation space uses the "decomposition" of $\mathcal{T}_N^*/N$ into nonarchimedean polyhedra to reduce the calculations to those for the action of a *finite* group on the nonarchimedean Riemann sphere.

## References


[1] Yves André, *Period Mappings and Differential Equations. From $\mathbb{C}$ to $\mathbb{C}_p$*, MSJ Memoirs, volume 12, Mathematical Society of Japan, Tokyo, 2003. Tôhoku-Hokkaidô lectures in arithmetic geometry, with appendices by F. Kato and N. Tsuzuki.

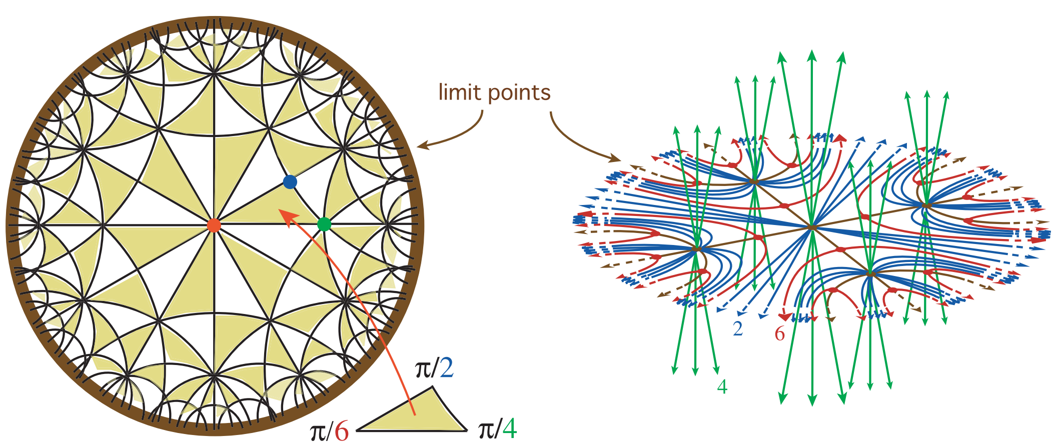




**Figure 16. A triangle group** $\Delta(2,4,6)$**.**

[2] John Baez, This week's finds in mathematical physics (week 62), August 28, 1995, `math.ucr.edu/home/baez/week62.html`.

[3] Gunther Cornelissen and Fumiharu Kato, Equivariant deformation of Mumford curves and of ordinary curves in positive characteristic, *Duke Math. J.* **116** (2003), 431- 470.

[4] Gunther Cornelissen, Fumiharu Kato, and Aristides Kontogeorgis, Discontinuous groups in positive characteristic and automorphisms of Mumford curves, *Math. Ann.* **320** (2001), 55–85.

[5] Lothar Gerritzen and Marius van der Put, *Schottky Groups and Mumford Curves*, Lecture Notes Math., volume 817, Springer, Berlin, 1980.

[6] A. Karrass, A. Pietrowski, and D. Solitar, Finite and infinite cyclic extensions of free groups, *J. Austral. Math. Soc.* **16** (1973), 458–466.

[7] Fumiharu Kato, Mumford curves in a specialized pencil of sextics, *Manuscripta Math.* **104** (2001), 451–458.

[8] ____ , Non-archimedean orbifolds covered by Mumford curves, *J. Algebraic Geom.* **14** (2005), 1–34.

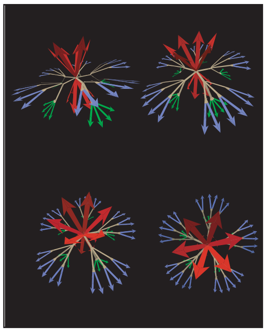
[9] Jean-Pierre Serre, *Trees*, Springer Monogr. Math., Springer-Verlag, Berlin, 2003.

## About the Cover

**Dyadic icosahedra**

The cover for this month exhibits fanciful renderings of the dyadic icosahedra discussed in the article by Gunther Cornelissen and Fumiharu Kato.

—*Bill Casselman*
*Graphics Editor*
(`notices-covers@ams.org`)



## Where Do *Notices* Covers Come From?

Usually *Notices* covers come from the articles inside the *Notices*. In practice, most covers are created specifically to relate to articles that have been accepted for publication. Sometimes the cover image will come straight from the article, but more often it takes off from an idea in the article, even sometimes a mere hint. But it is possible to go the other way—occasionally articles have been written to go with an idea for a cover.

If you have an image that you think would make a good cover, before submitting it you should write an article to go with the cover. The article should be of sufficient quality to be accepted on its own. But it need not be long; in principle, it would be reasonable for the article to simply explain the picture.

The simplest criterion to keep in mind is that the picture should try to explain something mathematical. If the picture is striking enough that it could stand on its own, then the accompanying article could explain the picture, rather than the picture illustrating the article.

—*Bill Casselman, Graphics Editor,*
(`notices-covers@ams.org`)