Manifolds with Density

Frank Morgan

Introduction

We consider a Riemannian manifold M^n with a positive density function $\Psi(x)$ used to weight volume and hypersurface area. In terms of the underlying Riemannian volume dV_0 and area dA_0, the new, weighted volume and area are given by

$$dV = \Psi dV_0, \quad dA = \Psi dA_0.$$

Such a density is not equivalent to scaling the metric conformally by a factor $\lambda(x)$, since in that case volume and area would scale by different powers of λ. Manifolds with density long have arisen on an ad hoc basis in mathematics. Quotients of Riemannian manifolds are manifolds with density. For example, $\mathbf{R}^3$ modulo rotation about the z-axis is the half-plane

$$H = \{(x, z) \colon x \geq 0\}$$

with density $2\pi x$; volume and area in $\mathbf{R}^3$ are given by integrating this density over the generating region or curves in H. A manifold with density of much interest to probabilists is *Gauss space* G^n: Euclidean space with Gaussian probability density

$$\Phi = (\gamma/2\pi)^{n/2} e^{-\gamma x^2/2}$$

(see e.g. [LT] or [S]; or [Bo1], [Bo2] for applications to Brownian motion and to stock option pricing). Different values of γ arise from scaling the metric and renormalizing to unit volume.

Manifolds with density merit further study. Gromov [G2] studies manifolds with density $\Psi = e^{\psi}$ as "mm spaces" and mentions the natural generalization of mean curvature

$$H_\psi = H - \frac{1}{n-1}\frac{d\psi}{dn}, \tag{1}$$

corresponding to the first variation of weighted area (see Proposition 10). There are various useful generalizations of Ricci curvature (see Bayle [Bay1] and references therein to Bakry, Émery, Ledoux, and others), generally involving Hess ψ and $d\psi \otimes d\psi$. My favorite generalization of Ricci curvature is simply

$$\mathrm{Ric}_\psi = \mathrm{Ric} - \mathrm{Hess}\,\psi, \tag{2}$$

the generalized curvature of Bakry-Émery [BE, Prop. 3] and Bakry-Ledoux [BL, p. 265]. For Gauss space G^n with density $\Phi = e^{\varphi}$ it is constant:

$$\mathrm{Ric}_\varphi = 0 - \mathrm{Hess}\,\varphi = \gamma I.$$

For a 2D Riemannian manifold with density $\Psi = e^{\psi}$, Corwin et al. [CHSX] use such a generalized Gauss curvature,

$$G_\psi = G - \Delta\psi,$$

and obtain a generalization of the Gauss-Bonnet formula for a smooth disc R:

$$\int_R G_\psi + \int_{\partial R} \kappa_\psi = 2\pi,$$

where κ_ψ is generalized curvature as in (1) and the integrals are with respect to unweighted Riemannian area and arclength.

Different generalizations of Gauss curvature, involving $|\nabla\psi|^2$, are needed to recover asymptotic formulas for areas and perimeters of small discs [CHSX, Props. 5.17 and 5.18].

Heintze-Karcher

In this note we present after Bakry-Ledoux, Bayle, and others (see [BL], [Bay1]) generalizations to manifolds with density $\Psi = e^{\psi}$ of the Heintze-Karcher volume estimate and the Levy-Gromov isoperimetric inequality. Heintze-Karcher [HK] provides an upper bound on the volume of a one-sided neighborhood of a hypersurface in terms of

Frank Morgan is professor of mathematics at Williams College. His email address is `Frank.Morgan@williams.edu`.

its mean curvature and the Ricci curvature of the ambient manifold. Our first, sharper generalization of Heintze-Karcher (Theorem 1) requires separate lower bounds on Ric and –Hess ψ. Our second generalization of Heintze-Karcher (Theorem 2) requires a single lower bound on the generalized Ricci curvature Ric_ψ of formula (2) above. Corollary 4 deduces a weak generalization of the theorem of Myers.

Levy-Gromov

The standard Levy-Gromov isoperimetric inequality, for a (compact) Riemannian manifold M with Ricci curvature bounded below by a positive constant δ, can be more easily stated after M and the comparison sphere of constant curvature δ are given constant densities with unit volume. After such renormalization, it says that M's isoperimetric profile $P(V)$ (least perimeter to enclose given volume) is greater than or equal to the comparison sphere's. Our generalization of the Levy-Gromov inequality to manifolds with variable density (Theorem 5) says that if the generalized Ricci curvature is bounded below by $\gamma > 0$,

$$\mathrm{Ric}_\psi = \mathrm{Ric} - \mathrm{Hess}\,\psi \geq \gamma > 0,$$

then M's isoperimetric profile satisfies

$$P(V) \geq P_G(V),$$

where $P_G(V)$ is the isoperimetric profile of *Gauss space* with constant generalized Ricci curvature γ.

Other Approaches

Finally, following Bayle [Bay1], [Bay2], we present first and second variation formulas, which provide an alternative, direct approach to isoperimetric inequalities. The related, more abstract, original approach of Bakry and Ledoux [BL] used Markov semigroup arguments. Our approach via Heintze-Karcher seems to be the simplest.

Acknowledgments

We acknowledge inspiring conversations with Ros [Ros], Bayle [Bay1], [Bay2], Tapp, and many students, including Joe Corneli, Neil Hoffman, and the 2004 Geometry Group [CHSX], at the 2001 Clay Research Institution Summer School at MSRI, in Granada, at the 2004 Minimal Surfaces School at Jussieu, Paris, and at Williams College. We thank the *Mathematical Intelligencer* editor and referee for helpful suggestions. This work is partially supported by grants from the National Science Foundation.

Isoperimetric Regions in Manifolds with Density

The isoperimetric problem in a smooth, n-dimensional Riemannian manifold with smooth density seeks a region of prescribed (weighted) volume of least (weighted) perimeter. If the total volume is finite so that no volume can disappear to infinity in the limit, the existence of an isoperimetric region with prescribed volume follows from standard compactness arguments of geometric measure theory [M1]. Furthermore, standard regularity applies [M2, 3.10]: the boundary of an isoperimetric region is a smooth submanifold except for a singular set of dimension at most $n-8$. Vanishing first variation implies constant generalized curvature (see Prop. 7).

The Estimate of Heintze and Karcher

For a closed hypersurface S in a Riemannian manifold M, the useful theorem of Heintze and Karcher [HK, Thm. 2.1], [BuZ, 34.1.10(1)] bounds the volume of a one-sided neighborhood of S in terms of the mean curvature of S and a lower bound on the Ricci curvature of M. The theorem has the following generalization to manifolds M with density. If the density is constant, formula (3) recovers the classical case as the exponential term reduces to unity. The proof is an easy modification of the classical case, as we will explain.

Theorem 1 *(Generalized Heintze-Karcher I). Let M^n be a smooth, complete Riemannian manifold with smooth density $\Psi = e^\psi$. Suppose that the Ricci curvature and density satisfy*

$$\mathrm{Ric} \geq (n-1)\delta$$

and

$$-\mathrm{Hess}\,\psi \geq \gamma.$$

Let S be a smooth, oriented, finite-area hypersurface in M with classical mean curvature $H(s)$. Let $V(r)$ denote the volume of the region within distance r of S on the side of the unit normal (which determines the sign of H). Then

$$V(r) \leq \int_S \int_0^{r^*(s)} [c_\delta(t) - H(s)s_\delta(t)]^{n-1} \times \exp\left(t\frac{d\psi}{dn}(s) - \gamma t^2/2\right)dt\,ds, \tag{3}$$

where ds denotes weighted surface area,

$$s_\delta(t) = \begin{cases} \delta^{-1/2}\sin\delta^{1/2}t & \text{for } \delta > 0, \\ t & \text{for } \delta = 0, \\ |\delta|^{-1/2}\sinh|\delta|^{1/2}t & \text{for } \delta < 0, \end{cases}$$

$c_\delta(t) = ds_\delta(t)/dt$, and r^* is the lesser of r and the first zero of $c_\delta(t) - H(s)s_\delta(t)$.

If equality holds, then S is umbilic, the region has constant curvature δ, and inside the region, along geodesics normal to S, $-d^2\psi/dt^2 = \gamma$.

Remark. Theorem 1 is sharp for hyperspheres and hyperplanes in Gauss space, where $-\mathrm{Hess}\,\psi = \gamma$, as well as for umbilic surfaces in Riemannian manifolds with constant density and constant curvature. The result generalizes to closed surfaces of higher codimension. Bayle ([Bay1], corrected and extended in [Bay2]), provides earlier alternative versions with an alternative hypothesis on ψ involving $d\psi$ and with the hypotheses on the Ricci curvature and the density combined into a single hypothesis, as in our Theorem 2.

Proof. When Ψ is constant, this is the standard Heintze-Karcher estimate, proved by integrating over infinitesimal normal wedges from S. (Every point in $V(r)$ is covered by the wedge from the nearest point of S and perhaps others as well.) Since $\text{Hess}\,\psi \le -\gamma$, at a point a distance t along a geodesic normal to S at s,

$$\psi(s,t) \le \psi(s,0) + t\frac{d\psi}{dn}(s,0) - \gamma t^2/2,$$

$$\Psi(s,t) \le \exp\left(t\frac{d\psi}{dn}(s,0) - \gamma t^2/2\right)\Psi(s,0),$$

with equality only if $d^2\psi/dt^2 = -\gamma$. The only change in the proof from the classical case is the introduction of this exponential factor.

By sacrificing some sharpness, Theorem 2 combines Ricci curvature and density in hypothesis and conclusion. This time we present a proof from scratch, which is easier than the standard proof of Heintze-Karcher for unit density.

Theorem 2 *(Generalized Heintze-Karcher II). Let M^n be a smooth, complete Riemannian manifold with smooth density $\Psi = e^\psi$ satisfying*

$$\text{Ric}_\psi = \text{Ric} - \text{Hess}\,\psi \ge \gamma.$$

Let S be a smooth, oriented, finite-area hypersurface in M with generalized mean curvature

$$H_\psi(s) = H(s) - \frac{1}{n-1}\frac{d\psi}{dn}.$$

Let $V(r)$ denote the volume of the region within distance r of S on the side of the unit normal (which determines the sign of H_ψ). Then

$$(4)\quad V(r) \le \int_S\int_0^r \exp(-(n-1)H_\psi(s)t - \gamma t^2/2)\,dt\,ds,$$

where ds denotes weighted surface area.

If equality holds, then S is umbilic, the region is flat, and inside the region, along geodesics normal to S, $-d^2\psi/dt^2 = \gamma$.

Remark. Theorem 2 is sharp for hyperspheres and hyperplanes in Gauss space and for totally geodesic surfaces in flat Riemannian manifolds of constant density.

Proof. We begin with the case $\Psi = 1$. This case actually follows almost immediately from the standard Heintze-Karcher or from Theorem 1, but we want to incorporate variable Ricci curvature, as in (6) below. Consider the volume element $e^{f(s,t)}dt\,ds$, corresponding to an infinitesimal slice dt of an infinitesimal normal wedge from S. Then $e^{f(s,t)}ds$ represents an element of surface area parallel to S. By the first variation formula for example (cf. (10)), its derivative $f'e^f ds$ equals $-(n-1)He^f ds$, so that

$$f' = -(n-1)H.$$

Since $-(n-1)H' = \Pi^2 + \text{Ric}(n,n)$, where Π is the second fundamental form (see e.g. Remark 8),

$$f'' = -\Pi^2 - \text{Ric}(n,n) \le -\text{Ric}(n,n),$$

with equality only if Π^2 vanishes. Hence by Taylor's theorem,

$$(5)\quad f(s,t) \le -(n-1)H(s)t - \int_0^t \text{Ric}(n,n)dt.$$

Consequently, since every point of $V(r)$ is covered by the infinitesimal wedge from the nearest point of S,

$$(6)\quad \begin{aligned} V(r) \le \int_S\int_0^r \exp\Big(&-(n-1)H(s)t \\ &- \int_0^1 \tau\,\text{Ric}(n,n)d\tau\Big)dt\,ds_0 \end{aligned}$$

where we now write ds_0 to emphasize that this is the case of unweighted area.

For general density $\Psi = e^\psi$,

$$(7)\quad \psi(s,t) \le \psi(s,0) + t\frac{d\psi}{dn}(s,0) + \int_0^t \tau\frac{d^2\psi}{dn^2}d\tau.$$

Preparing to add f and ψ, note that

$$-(n-1)H(s) + t\frac{d\psi}{dn}(s) = -(n-1)H_\psi(s)$$

and that by hypothesis

$$\begin{aligned} &-\int_0^t \tau\,\text{Ric}(n,n)d\tau \\ &\qquad + \int_0^t \tau\frac{d^2\psi}{dn^2}d\tau \le -\gamma\int_0^t \tau\,d\tau = -\gamma t^2/2. \end{aligned}$$

Hence

$$f(s,t) + \psi(s,t) \le \psi(s,0) - (n-1)H_\psi(s)t - \gamma t^2/2.$$

Therefore

$$\begin{aligned} V(r) &\le \int_S\int_0^r e^f e^\psi\,dt\,ds_0 \\ &\le \int_S\int_0^r \exp(-(n-1)H_\psi(s)t - \gamma t^2/2)dt(e^{\psi(s,0)}ds_0), \end{aligned}$$

as desired.

Remark 3. Theorem 2 and its proof apply to perimeter minimizers S with singularities. Indeed, for any point off S, the nearest point on S is a regular point, because the tangent cone lies in a halfspace and hence must be a hyperplane.

The following immediate corollary of Theorem 2 provides a generalization of Myers's theorem [M3, 9.6], which says that a smooth, complete, connected Riemannian manifold with a positive lower bound on the Ricci curvature is compact (and provides an estimate on the diameter).

Corollary 4. *Let M^n be a smooth, complete, connected Riemannian manifold with smooth density $\Psi = e^{\psi}$ satisfying*

$$\mathrm{Ric}_{\psi} = \mathrm{Ric} - \mathrm{Hess}\,\psi \geq \gamma > 0.$$

Then M has finite volume.

Remark. It does not follow that M is compact, as shown for example by Gauss space. Nor is there any quantitative bound on the volume, since scaling the density scales the volume but leaves Ric_{ψ} unchanged.

The Isoperimetric Inequality of Levy and Gromov

Theorem 5 gives a generalization of the isoperimetric inequality of Levy and Gromov ([Gr1, 2.2], [BuZ, 34.3.2], or [Ros, Sect. 2.5]) to a manifold M with density, with the sphere replaced by Gauss space as the model of comparison. It includes the sharp isoperimetric inequality for Gauss space of Sudakov-Tsirel'son [ST] and Borell [Bo1]. If M has finite volume, we may assume by scaling the density that M has unit volume. Such scaling does not affect the generalized Ricci curvature (8), because multiplying the density e^{ψ} by a constant just adds a constant to ψ and leaves Hess ψ unchanged. As described in the introduction, such a normalization makes the statement of even the classical Levy-Gromov much more transparent.

Theorem 5 *(Generalized Levy-Gromov). Let M^n be a smooth, complete, connected Riemannian manifold with smooth density $\Psi = e^{\psi}$, unit volume, and generalized Ricci curvature*

$$\mathrm{Ric}_{\psi} = \mathrm{Ric} - \mathrm{Hess}\,\psi \geq \gamma > 0. \tag{8}$$

Then the isoperimetric profile $P(V)$ (least perimeter to enclose given volume) satisfies

$$P \geq P_{G_\gamma}, \tag{9}$$

where P_{G_γ} is the isoperimetric profile of Gauss space with density

$$\Phi = e^{\varphi} = (\gamma/2\pi)^{n/2} e^{-\gamma x^2/2},$$

so that $\mathrm{Ric}_{\varphi} = -\mathrm{Hess}\,\varphi = \gamma$.

In Gauss space, perimeter minimizers are hyperplanes. If equality holds in (9) *for some $0 < V < 1$, then M is a product of 1D Gauss space with some $(n-1)$D Euclidean space with density.*

Remark. P_{G_γ} is independent of dimension, because the volume of a hyperplane bounding given volume is independent of dimension, because Gauss space is a product of 1D Gauss spaces. Hence P_{G_γ} is just the value of the Gaussian density

$$\sqrt{\frac{\gamma}{2\pi}}\, e^{-\gamma x^2/2}$$

at the endpoint of a half-line with the given mass.

Proof. Note that the generalized curvature H_{φ} of the hyperplane $S_0 = \{x_1 = a\}$ in Gauss space is given by

$$H_{\varphi} = H - \frac{1}{n-1}\frac{d\varphi}{dn} = 0 - \frac{1}{n-1}\gamma a,$$

which is constant. For given $0 < V < 1$, let P be the perimeter of a minimizing hypersurface S in M, and let P_0 be the perimeter of the hyperplane S_0 in Gauss space. By replacing V by $1-V$ (which changes the sign of the mean curvatures) if necessary, we may assume that the generalized mean curvature of S is greater than or equal to that of S_0. By generalized Heintze-Karcher (Theorem 2 with Remark 3),

$$\frac{V}{P} \leq \frac{V}{P_0} = \int_0^{\infty} \exp(\gamma a t - \gamma t^2/2)dt.$$

Taking M to be Gauss space, we conclude that hyperplanes are perimeter minimizing. If equality holds, then equality holds in (4) for $r = \infty$ on both sides of S, S is umbilic with mean curvature $H = 0$, and hence S is a hyperplane. We conclude that in Gauss space, hyperplanes are uniquely perimeter minimizing.

Returning to general M, we conclude that $P \geq P_{G_\gamma}$. Suppose that equality holds for some $0 < V < 1$. Then S has the same generalized mean curvature as S_0, and equality in Theorem 2 holds on both sides of S (with $r = \infty$). Consequently, S is totally geodesic; M is Euclidean space (with some density); and along geodesics normal to S, $-d^2\psi/dt^2 = \gamma$. Since S has constant generalized mean curvature, on S, $d\psi/dn$ is constant. Therefore M is a product of 1D Gauss space with some $(n-1)$D Euclidean space with density.

As a corollary, we recover a nonsharp version of an isoperimetric inequality of Barthe.

Corollary 6 *[Bar, Prop. 11]. For an nD round sphere of radius and density to make the volume and equator area both 1,*

$$P/P_{G_{2\pi}} > c_n = \sqrt{\frac{n-1}{2}}\,\frac{\frac{n-2}{2}!}{\frac{n-1}{2}!},$$

where $x!$ *means* $\Gamma(x+1)$.

As n approaches infinity, c_n approaches Barthe's value of 1, which is sharp at $V = 1/2$.

Proof. Since the unit nD sphere has volume

$$V_n = \frac{(n+1)\pi^{(n+1)/2}}{((n+1)/2)!}$$

and equator area V_{n-1}, our sphere has Ricci curvature

$$\gamma = (n-1)(V_n/V_{n-1})^2.$$

By Theorem 5,

$$P > P_{G_\gamma} = \sqrt{\gamma/2\pi}\,P_{G_{2\pi}}.$$

Finally, note that

$$\sqrt{\gamma/2\pi} = \sqrt{\frac{n-1}{2\pi}}\frac{v_n}{V_{n-1}} = \sqrt{\frac{n-1}{2}}\frac{\frac{n+1}{2}}{\frac{n}{2}}\frac{\frac{n}{2}!}{\frac{n+1}{2}!} = \sqrt{\frac{n-1}{2}}\frac{\frac{n-2}{2}!}{\frac{n-1}{2}!}.$$

First and Second Variation

An alternative approach to isoperimetric inequalities, and the one followed by Bayle [Bay1], [Bay2], uses just second variation. For the record, we present such formulas for manifolds with density $\Psi = e^\psi$. The classical formulas are augmented by terms involving the first and second normal derivatives of ψ.

Proposition 7 *[Bay1, Sect. 3.4.6]. Let M^n be a smooth Riemannian manifold with smooth density $\Psi = e^\psi$. Let S be a smooth hypersurface, and consider a smooth normal variation of compact support of constant velocity $u(s)$ along the geodesic normal to S at s. Then the first and second variations of (weighted) area satisfy*

$$\delta^1(u) = -\int_S u(n-1)H_\psi, \tag{10}$$

$$\delta^2(u) = \int_S |\nabla u|^2 + u^2(n-1)^2H_\psi^2 - u^2\Pi^2 - u^2\mathrm{Ric}(n,n) + u^2(d^2\psi/dn^2), \tag{11}$$

where Π is the second fundamental form and the generalized mean curvature H_ψ satisfies

$$H_\psi = H - \frac{1}{n-1}\frac{d\psi}{dn}.$$

Remark 8. If $u = 1$, then the second variational formula is equivalent to the first, plus the fact that

$$(n-1)\frac{dH}{dt} = \Pi^2 + \mathrm{Ric}(n,n),$$

which follows from the case of the curvature κ of a curve in a surface of Gauss curvature G:

$$\frac{d\kappa}{dt} = \kappa^2 + G.$$

The following corollary of Proposition 7 on the isoperimetric profile $P(V)$ (least perimeter for given volume) follows from the fact that $P(V+\Delta V)$ is at most the perimeter of a uniform perturbation of the minimizer for volume V, as in [MJ, 2.1] (it turns out that the singularities of S are negligible).

Corollary 9 *[Bay1, (3.40)]. Let M^n be a smooth, complete, finite-volume Riemannian manifold with smooth density $\Psi = e^\psi$ with*

$$\mathrm{Ric} - \mathrm{Hess}\,\psi \geq \gamma.$$

Then the isoperimetric profile P and its derivatives P', P'' satisfy

$$PP'' \leq -\frac{P'^2}{n-1} - \gamma \tag{12}$$

almost everywhere (and in a weak sense everywhere). If equality holds, then a perimeter minimizer is flat.

Remark 10. By scaling the density, which does not affect $\mathrm{Hess}\,\psi$ or PP'' or hence the hypothesis or conclusion of Corollary 9, one may assume that M has unit volume. Since equally holds in (12) for Gauss space, isoperimetric estimates such as Levy-Gromov (Theorem 5) follow.

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