Book Review

Mathematics by Experiment and *Experimentation in Mathematics*

Reviewed by Jeffrey Shallit

Mathematics by Experiment
Jonathan Borwein and David Bailey
A K Peters, 2003
288 pages, $45.00
ISBN 1-56881-211-6

Experimentation in Mathematics
Jonathan Borwein, David Bailey, and
Roland Girgensohn
A K Peters, 2004
357 pages, $49.00
ISBN 1-56881-136-5

Is mathematics an experimental science? And is the computer the microscope (or the telescope) of mathematics? As a young mathematician I was certainly convinced this was so. Influenced by Kenneth Iverson (1920–2004), the inventor of the computer language APL, I saw the computer as an experimental tool that would reveal new mathematical worlds. As Iverson and others wrote in 1970, with the use of a computer, "mathematics becomes a laboratory science, open to experiment, conjecture, and discovery" [1].

Inspired by a 1972 paper of Lang and Trotter [3], as a teenager I wrote programs to compute the continued fraction expansion of various algebraic numbers. After verifying their results, I was naturally led to compute the expansions of other real numbers. One of the numbers I tried was

$$\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^8} + \cdots,$$

and I was astonished to find that all the partial quotients (i.e., the terms of the continued fraction) were either $1, 2, 4$, or 6. I then tried other numbers of the same form and found similar behavior. Of course, this demanded an explanation. I eventually found one, and this led to my first serious published paper [5].

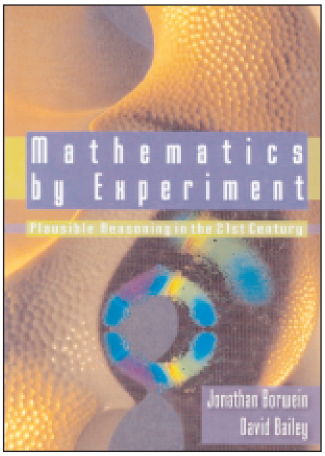
In this I was following in famous footsteps. Gauss, for example, was led to conjecture the prime number theorem by studying the distribution of primes in published tables. It is clear that experimentation with examples is an important part of the mathematician's toolbox. And the computer allows experimentation far beyond the range of hand calculation.

Yet there was—and still is—resistance to the computer as a tool. Tymoczko, for example, suggested that the computer-aided solution of the four-color conjecture in graph theory introduced a new and fundamentally different form of unreliability in mathematical proof [6], [7]. At a recent conference, when I presented a result on avoidability in combinatorics on words that depended on a large calculation by computer, a colleague expressed his dissatisfaction that he could not verify my theorem entirely by hand. Yet why should every simple theorem have a simple proof?

Even today, with nearly universal access to computers, many students are unfamiliar (or uncomfortable) with the experimental approach. When I ask my students whether the decimal expansion of $\sqrt{3}$ contains three consecutive identical digits, many are completely stumped how to even begin to attack the problem. The idea that they should use a computer to find the first hundred or thousand digits does not occur to them.

So, despite the wide availability of computers, the experimental approach needs advocates, and Jonathan Borwein and David Bailey are happy to step in. In the two books under review (*Mathematics by Experiment* and *Experimentation in*

Jeffrey Shallit is professor of mathematics in the School of Computer Science at the University of Waterloo. His email address is shallit@graceland.math.uwaterloo.ca.



Mathematics) they develop the value of this approach in grand style (in the second book they are joined by Roland Girgensohn).

In *Mathematics by Experiment* they define experimental mathematics to be "the methodology of doing mathematics that includes the use of computations for:

1. Gaining insight and intuition.
2. Discovering new patterns and relationships.
3. Using graphical displays to suggest underlying mathematical principles.
4. Testing and especially falsifying conjectures.
5. Exploring a possible result to see if it is worth formal proof.
6. Suggesting approaches for formal proof.
7. Replacing lengthy hand derivations with computer-based derivations.
8. Confirming analytically derived results."

The two books are quite similar in scope, but *Mathematics by Experiment* is more introductory in nature. *Experimentation in Mathematics* covers some of the same material but is longer (357 pages versus 288 pages) and takes a deeper, less conversational approach. Both books emphasize areas where experimental mathematics has been most successful: number theory, algebra, and combinatorics.

Mathematics by Experiment covers a wide variety of topics: evaluation of definite integrals, evaluation of infinite series, the $3x + 1$ problem, simplification of radicals, dilogarithms, hypergeometric functions, the calculation of π, normality of real numbers, the fundamental theorem of algebra, the gamma function, Stirling's formula, the arithmetic-geometric mean, arbitrary precision arithmetic, and integer relation algorithms. Along the way we get song lyrics by Tom Lehrer; pictures of sculptures by Helaman Ferguson; and entertaining quotations from Hardy, Feynman, Milnor, Darwin, Thurston, and Keynes. Although undisciplined at times, it is a book that can be enjoyed by undergraduates and professional mathematicians alike.

For deeper applications the reader will want to continue with *Experimentation in Mathematics*. Here the reader will find chapters entitled "Sequences, Series, Products and Integrals", "Fourier Series and Integrals", "Zeta Functions and Multizeta Functions", "Partitions and Powers", "Primes and Polynomials", "The Power of Constructive Proofs II", and "Numerical Techniques II". There is also more emphasis on theorems and proofs.

To illustrate the game, let's look at two basic tools of experimental mathematics: *sequence recognition* and *real number* recognition.

Sequence recognition comes in handy when we are given a sequence $(a(n))_{n\geq 1}$ defined by a summation formula and we want to find a simpler expression for it, perhaps in "closed form". The traditional mathematical approach would be to examine the definition for $a(n)$ and manipulate it in some way, perhaps using familiar tools such as binomial coefficient identities, changing the order of summation, etc. An experimental mathematician, however, will simply compute the first ten or so values of $a(n)$ and then look up the result in Neil Sloane's "On-Line Encyclopedia of Integer Sequences", available at `http://www.research.att.com/~njas/sequences/`. With luck such a search will produce a known closed form and half a dozen citations to the literature where the sequence's properties are discussed. All that is left to do (!) is prove that our expression is, indeed, identical to the known representation. Here a symbolic algebra system, such as Maple, often proves useful. Depending on the problem domain, special-purpose tools, such as the Wilf-Zeilberger algorithm, can actually prove our result for us.

This process is illustrated in section 2.2 of *Mathematics by Experiment*. Borwein and Bailey discuss the observation that if Gregory's series for π,

$$4\sum_{k\geq 1}\frac{(-1)^{k+1}}{2k-1},$$

is truncated after 5,000,000 terms, then the decimal expansion of the result agrees with π at many places, with exceptions occurring with a period of 14. When one examines the coefficients corresponding to the errors at these positions, one finds the coefficients are $2, -2, 10, -122, 2770, \ldots$. Dividing by 2 and searching Sloane's table produces the guess that these are the Euler numbers, and indeed one can then find an asymptotic expansion for

$$\frac{\pi}{2} - 2\sum_{1\leq k\leq N/2}\frac{(-1)^{k+1}}{2k-1}$$

involving the Euler numbers.

Constant recognition is similar. How, for example, could we evaluate

$$\sum_{k\geq 1}\left(\frac{k^k}{k!e^k} - \frac{1}{\sqrt{2\pi k}}\right)?$$

An experimental mathematician might simply compute the sum to twenty digits and then use a "number recognizer" (such as that at `http://www.cecm.sfu.ca/projects/ISC`) to find that the sum appears to be

$$-\frac{2}{3} - \frac{1}{\sqrt{2\pi}}\zeta\left(\frac{1}{2}\right).$$

Once the form of the result is suspected, a proof follows (aided by the symbolic algebra system Maple).

Although I found both books very entertaining, they each show some signs of being put together too hastily. Sometimes terms are used before they are defined. For example, on page 24 of *Mathematics by Experiment* Borwein and Bailey present a collection of ten interesting challenges in experimental mathematics. While most will be comprehensible to bright undergraduates in mathematics, the very first says, "Compute the value of r for which the chaotic iteration $x_{n+1} = rx_n(1 - x_n)$, starting with some $x_0 \in (0, 1)$, exhibits a bifurcation between 4-way periodicity and 8-way periodicity." There is no explanation of the meaning of this technical jargon, and the reader has to wait until page 51 to find one.

In the same book, pages 56 and 248 both contain very similar accounts of the discovery that

$$\sum_{k=1}^{\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{k}\right)^2 k^{-2} = \frac{17\pi^4}{360}.$$

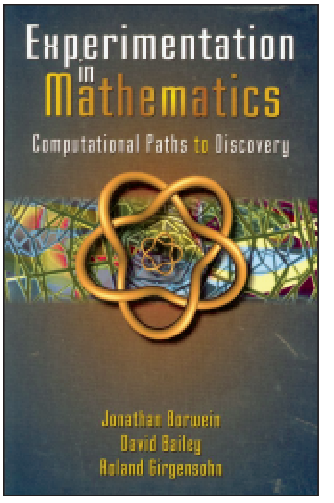
And Section 1.8.1 of *Experimentation in Mathematics* reprises Section 2.2 of *Mathematics by Experiment* without adding anything really new (and mistakenly calls it Section 1.3). More careful editing would have removed these redundancies.

Also, the scholarship is not as good as it could be. The authors do not adequately address the influence of experimental pioneers such as Derrick Lehmer [4] or Horst Zimmer [8]. In fact, none of the papers in the bibliography below are cited in either of the two books.

As I said at the beginning of this review, as a young man I believed strongly in the gospel of experimental mathematics. And it is certainly true that this approach has led to dozens of interesting new directions in combinatorics, number theory, and algebra. Do I still believe? Yes. Experimental mathematics is, and will continue to be, very fruitful. But let me offer three caveats.

First, mindless computation can be counterproductive. I often see queries on electronic mailing lists devoted to mathematics of the form "I wrote a program to verify the following property for the first billion integers. Is it always true?" immediately answered by someone else who gives a one-line proof of the property. Time used thinking—away from the computer—is often time well spent. As H. H. Williams remarked, "Furious activity is no substitute for understanding."

Second, naive computation can lead to incorrect conjectures. For example, computers typically (but not always) represent real numbers using floating-point numbers, and careless computation with these approximations can have surprisingly bad results—a fact well known to numerical analysts for years [2]. To their credit, Borwein, Bailey, and Girgensohn recognize this and even give some entertaining examples in a section of *Experimentation in Mathematics* entitled "High Precision Fraud". Does



$$\sum_{n\geq 1} 10^{-n}\lfloor n\tanh(\pi)\rfloor$$

really equal $1/81$? No, but you won't find the answer by computing the first two hundred digits.

Third, experimental mathematics has its limits. Experimental mathematics probably would not have led to a proof of Gödel's theorem or the Poincaré conjecture. And how, for example, can it be fruitfully used in Kolmogorov complexity, where the objects under discussion are often uncomputable in a formal sense?

Still, experimental mathematics is here to stay. The reader who wants to get an introduction to this exciting approach to doing mathematics can do no better than these interesting books.

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