Knots for Everyone: *The Knot Book*

Reviewed by Alexey Sossinsky

The Knot Book: An Elementary Introduction to the Mathematical Theory of Knots
Colin C. Adams
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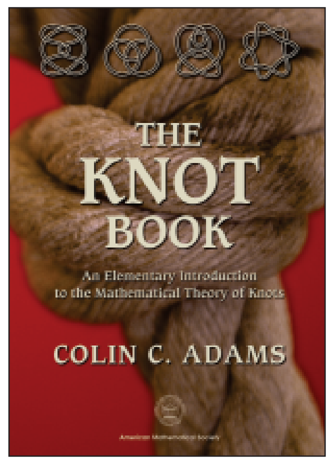
Knot theory has been very fortunate with books—from the first one, Kurt Reidemeister's basic and elegant *Knottentheorie* (1928); to the small *Introduction to Knot Theory* by R. Crowell and R. Fox (1963), with its beautiful exposition of the Alexander polynomial based on the free differential calculus; to Dale Rolfsen's superb *Knots and Links* (1976),[1] where many of us learned knot theory; to the fundamental *Knots* by W. Burde and H. Zieschang (1984), with a wealth of material summarizing the pre-Vaughan-Jones period in the theory; to L. Kauffman's *Knots and Physics* (1992), brimming with unexpected ideas and interconnections—to name only the first five titles that come to mind.[2] Nevertheless, the book under review stands out even in this elite company and deserves the definite article in its title: this is indeed THE knot book.

The Knot Book is addressed to the nonspecialist, requires practically no knowledge of any serious college math, and can be read with ease and pleasure by any undergraduate interested in mathematics. In fact, it can give rise to such an interest, showing as it does the beauty and depth of what may be called "living mathematics", and can convince the beginner that math is not just a set of dogmatic rules that one is forced to learn because they are useful, that "doing mathematics" can be just as creative and exciting as writing music or designing bridges.



At the same time, the professional research mathematician (and even experts in the field) will find the book equally rewarding: it is densely packed with facts and ideas, clearly explains the motivation of ongoing research, cleverly avoids the technically difficult places or succeeds in giving clear and simple explanations in situations that had previously seemed very intricate to some,[3] and lists a large number of open problems.

Originally, the book was published in 1994, and the American Mathematical Society's edition is an exact page-by-page replica of the original text and illustrations. The only difference is that a page of corrections to the first edition has been added.

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[1] *Originally published by Publish or Perish, it is now available in a new AMS edition.*

[2] *Of course, there are more books on knot theory worthy of mention: in particular, the recent ones by V. Jones, A. Kawauchi, K. Murasugi, W. Lickorish, and C. Livingston, but those listed above were the first to come to my mind.*

[3] *For example, this reviewer, who has known and used the Arf invariant during his whole mathematical life, finally understood what it really means upon reading the short text devoted to it in the book.*

The early 1990s was an excellent period for writing and publishing a book on knot theory. Indeed, the main content of knot theory before 1985 had just been systematically described in the Burde-Zieschang book when the field suddenly exploded after Vaughan Jones's discovery of his famous polynomial and the subsequent research of those who followed him: Edward Witten, Vladimir Drinfeld, Maxim Kontsevich, to mention only the Fields Medallists. Perhaps even more important (if unrewarded) was the seminal work of Victor Vassiliev. All of the above-mentioned research required some very sophisticated mathematics. Colin Adams would hardly have succeeded in giving an elementary treatment of knot polynomials if Conway had not drastically simplified the theory of the Alexander polynomial and if Louis Kauffman had not unexpectedly come up with his simple and extremely original approach to the Jones polynomial.

One may wonder why the book was reprinted without any changes ten years later. Couldn't the exposition of the book's material be improved in the light of the past decade? Shouldn't new material be added? After the flurry of brilliant research in the late 1980s and early 1990s, had the flow of results in the field suddenly gone dry? My answer to the first question is a resolute NO. Of course there is no such thing as ultimate perfection in any textbook, but this one is so solidly composed and so well written that I see no need for any rewriting. And although knot theory has continued to develop with great intensity, nothing has happened in the last decade that would warrant changing the exposition of the subject matter chosen by the author.

Concerning the new material developed in the past decade, much of it is far from elementary and therefore not suited to the style of Adams's book. However, I seriously regret that some of the new results have not been added. Several have very simple and important formulations and would fit nicely in the book. For example, the result of Joel Hass and Jeff Lagarias asserting that there is an upper bound on the number of Reidemeister moves required for unknotting would fit well in the first introductory chapter (without proof, of course). Other results having a strong geometric flavor and not involving any intricate mathematics, e.g., Ivan Dynnikov's work on three-page books and unknotting algorithms or Louis Kauffman's results on virtual knots, are conceptually quite elementary and would be appropriate as additional material to the book. Of course these are not the most important knot theory results of the past decade, just some new developments that would be in the spirit of Colin Adams's book—i.e., can be explained to the undergraduate math major—nor are they the only ones of that type, just some that came to mind.

Now let us look at the structure of the book, summarize its contents, and, as we go along, point out some of its characteristic stylistic aspects.

The first chapter is an introduction that goes very far to the heart of the subject. It includes Reidemeister moves, knot composition, and the first nontrivial practically computable example of a knot invariant (tricolorability). All of this is explained with remarkable simplicity and visual clarity (there is at least one figure on each page of the introduction). An important feature of this chapter is the large number of exercises (thirty-seven in all, most of them quite simple, usually meant to be solved by manipulations with strings, electric cords, or on paper with pencil and eraser). There are no less than six unsolved questions (problems) in the thirty pages of the introduction, and, surprisingly, they sound just as simple as the exercises, immediately demonstrating to the beginner that knot theory is very much alive and full of challenges and reminding the expert about many simple questions to which we still have no answers.

The second chapter is about how it all started (with the knot tables of Tait and Little) and what has been achieved since in the tabulation of knots (mostly by Morwen Thistlethwaite's computer). The exposition includes Hugh Dowker's and John Conway's computer-friendly notation for knots (the latter involving tangles). Here again there are several unsolved questions and numerous exercises (thirty of them). The chapter begins with a hilarious quotation from a nineteenth-century paper by the Reverend Thomas Kirkman (politely described by Colin Adams as due to the latter's "opaque writing style"); contains a very transparent description of the beautiful relationship between the construction of rational tangles and continued fractions (nothing opaque about Adams's writing style!); and ends with the encoding of alternating knots by graphs (via checkerboard coloring), a neat geometric construction now used in statistical physics.

The third chapter, called "Invariants of Knots", does not deal with knot polynomials (as the expert would expect), but with three extremely simple but hopelessly uncomputable (from the practical[4] point of view) invariants: namely, the crossing number, the bridge number, and the unknotting number. (Actually, there is a practically implementable algorithm for computing the crossing number, based on polynomial invariants and explained in one of the book's concluding chapters.)

Chapter 4 is about surfaces and their relationship with knots. Before going on to the expected

[4]*Theoretically, these invariants* ***are*** *computable, since the knot classification problem is algorithmically decidable according to Serguei Matveev; see his book* Algorithmic Topology of 3-Manifolds, *Springer-Verlag, 2004.*

topic of Seifert surfaces, the chapter begins with a very visual account of topological surfaces, giving the beginner a clear understanding of the intrinsic topology of a surface (the difference between homeomorphism and isotopy is explained) and a number of ideas underlying the proof of the classification theorem of compact 2-manifolds (triangulation, connected sum, orientability, the Euler characteristic). However, no proof of the classification theorem appears; in fact, the theorem is not even stated. This is in keeping with the style of the book: the definition-theorem-proof kind of exposition, standard in most math textbooks, first appears in this chapter (the fourth!) and is used very sparingly further; there are no long or complicated proofs at all. (It should be noted that in proofs and other arguments, many intuitively clear facts are not proved but used, e.g., the Jordan curve theorem and general position in this chapter.) After a glance at surfaces with boundary, the chapter concludes with the construction of the Seifert surface of a knot and the notion of genus.

In the next, fifth, chapter, Colin Adams undertakes a seemingly impossible task: to explain, in an elementary way to people lacking the necessary prerequisites, what a hyperbolic knot and a hyperbolic manifold are, and what curvature and volume are. This reviewer, who has always regarded Thurston's work with pious admiration, would never have thought this could be done. But Adams performs this *tour de force* with remarkable clarity and in simple, everyday words. In passing, he mentions Jeff Weeks's amazing SNAPPEA software (which computes the volumes of complements to hyperbolic knots) and lists a number of simply formulated but extremely difficult unsolved problems. The exposition then unexpectedly shifts to braids (as a way of constructing knots and links), goes on to essentially explain Artin's theorem (without stating or proving or even naming it), and concludes with the Markov theorem (stated and explained, but not proved). The chapter ends with a discussion of almost alternating knots, about which several unsolved questions appear.

The sixth chapter is central to the book and gives a clear exposition of the Jones polynomial (via the Kauffman bracket), as well as of the Alexander and HOMFLY polynomials (via the appropriate skein relations). It also describes the Jones polynomial for alternating knots, which leads up to the Kauffman-Murasugi-Thistlethwaite theorem justifying Tait's hundred-year-old conjectures about alternating knot projections. This theorem is actually rigorously proved in the text.

I will not describe the contents of the fascinating seventh chapter, devoted to applications to biology and connections with various branches of physics. Read it, if nothing else in the book.

Chapter 8 is mostly about graphs and explains the beautiful Conway-Gordon theorem about the appearance of linked circles or nontrivial knots (Hamiltonian cycles) in the embedding of certain nonplanar graphs (with almost complete proofs). Then there is a discussion of the famous dichromatic polynomial and its relationship with the link polynomials described in Chapter 6 and—this is one of the most striking connections of knot theory and physics—with the Potts model (two-dimensional water-ice).

The last two chapters, entitled "Topology" and "Higher Dimensional Knotting", are not about classical knots (in 3D), but still contain a lot of beautiful and relevant information.

There is an appendix with knot and link tables (up to nine and seven crossings respectively) supplied with the values of their standard invariants.

The bibliography is organized by chapters, and the references are supplied with Colin Adams's brief comments, intended to clarify their contents to the reader. This is very convenient for the beginner and rather annoying for the expert (it takes a while to find any given reference unless you correctly guess what chapter it pertains to).

Any self-respecting reviewer is expected to demonstrate his competence and meticulousness by pointing out errors, misprints, and other defects of the book under review. This turned out to be practically impossible in this case: I was unable to find any of the aforementioned drawbacks in Colin Adams's book. The best I can do is to reiterate my regrets about the fact that none of the achievements of the last decade appear in the present edition and to note that many names important for knot theory are never mentioned in it. Thus, it seemed strange to me that, in the chapter on braids, neither Emil Artin nor Joan Birman is cited, and there is no mention at all in the book of four of the mathematical superstars who have contributed the most to knot theory: Vladimir Drinfeld, Maxim Kontsevich, Victor Vassiliev, and Edward Witten.

This is a beautiful, fascinating, and very readable book, written in a clear, down-to-earth, and no-nonsense style. It is, in my opinion, one of the best books of all time for introducing mathematics and for giving an in-depth idea of present-day research mathematics to the beginner. I am sure it will convince many young people that mathematics is alive and worth doing. For the professional mathematician, it contains a very accessible account of a huge amount of material about knot theory and can be used as a reference book even by the experts.

By all means, get this book and start reading it. But beware: you'll find it hard to stop once you do.