



a Quantum Group?

Shahn Majid

A quantum group is in the first place a remarkably nice object called a Hopf algebra, the axioms for which are so elegant that they were written down in the 1940s well before truly representative examples emerged from physics in the 1980s. So let us start with these elegant axioms, but with the caveat that it's the modern examples and their further structure that really make the subject what it is. A Hopf algebra H obeys the following axioms:

1. H is a unital algebra $(H, \cdot, 1)$ over a field k .
2. H is a counital coalgebra (H, Δ, ϵ) over k . Here the "coproduct" and "counit" maps $\Delta : H \rightarrow H \otimes H$ and $\epsilon : H \rightarrow k$ are required to obey $(\Delta \otimes \text{id})\Delta = (\text{id} \otimes \Delta)\Delta$ and $(\epsilon \otimes \text{id})\Delta = (\text{id} \otimes \epsilon)\Delta = \text{id}$.
3. Δ, ϵ are algebra homomorphisms.
4. There exists an "antipode" map $S : H \rightarrow H$ obeying $\cdot(\text{id} \otimes S)\Delta = \cdot(S \otimes \text{id})\Delta = 1\epsilon$.

There are three points of view leading independently to these axioms. Each of them defines what a quantum group is. For lack of space we will focus mainly on the first of these.

The first point of view starts with the observation that the functions $k(G)$ on a finite group or the coordinate algebra $k[G]$ of an algebraic group form Hopf algebras. For any finite set let $k(G)$ be the pointwise algebra of functions on G with values in k . We identify $k(G) \otimes k(G) = k(G \times G)$, i.e., functions in two variables. Then, when G is actually a group, we define for all $a \in k(G)$,

$$(\Delta a)(x, y) = a(xy), (Sa)(x) = a(x^{-1}), \epsilon(a) = a(e),$$

where e is the group unit element and $x, y \in G$ are arbitrary. We see that the group structure is encoded in the coalgebra Δ, ϵ and antipode S . Similarly, for every subset $G \subseteq k^n$ described by polynomial equations one has a "coordinate algebra" $k[G]$ defined as polynomial functions on k^n , modulo the ideal of functions that vanish on G . When k is algebraically closed we obtain in this way a precise (functorial) correspondence between such polynomial subsets and nilpotent-free commutative algebras with a finite set of generators. This is the basic setting of algebraic

Shahn Majid is professor of mathematics at Queen Mary, University of London. His email address is s.majid@qmul.ac.uk.

geometry. When the subset G forms a group and the group law is polynomial, the product map $G \times G \rightarrow G$ becomes under the correspondence an algebra homomorphism Δ going the other way. Likewise for the rest of the Hopf algebra structure. Two examples are as follows. The "affine line" is described by the coordinate algebra $k[x]$ (polynomials in one variable) with additive coproduct $\Delta x = x \otimes 1 + 1 \otimes x$ corresponding to addition in k . The reader can and should fill in the rest of the structure and verify that one has a Hopf algebra in fact for any field k . The "circle" is similarly described by the coordinate algebra $k[t, t^{-1}]$ (polynomials in t, t^{-1} with the implied relations $tt^{-1} = t^{-1}t = 1$) and multiplicative coproduct $\Delta t = t \otimes t$ corresponding to multiplication in k^* . Again, the reader should fill in and verify the rest of the Hopf algebra structure. Most familiar complex Lie groups are likewise defined by polynomial equations and have corresponding algebras $\mathbb{C}[G]$, as well as versions $k[G]$ defined over general fields with the same relations. Meanwhile, working over \mathbb{C} , a "real form" means the additional structure of a compatible complex-linear involution making the coordinate algebra into a $*$ -algebra. In this case one can denote the above two examples as $\mathbb{C}[\mathbb{R}]$ and $\mathbb{C}[S^1]$ when taken with $x^* = x$ and $t^* = t^{-1}$ respectively.

A general Hopf algebra H similarly has the structures Δ, ϵ, S but we do not assume that the algebra of H is commutative as it is in the above examples. This is the point of view of noncommutative geometry or "quantisation" in the mathematician's (but not physicist's) sense of a noncommutative deformation of a commutative coordinate or function algebra. Much of group theory and Lie group theory proceeds at this level; for example a translation-invariant integral $\int : H \rightarrow k$ (in a certain sense involving Δ), if it exists, is unique up to scale and does indeed exist in nice cases. Likewise the notion of a complex of differential forms $(\oplus_n \Omega^n, d)$ makes sense over any algebra H . At degree 1 the space Ω^1 of 1-forms is required to be an $H - H$ bimodule equipped with an operation $d : H \rightarrow \Omega^1$ obeying the Leibniz rule

$$d(ab) = (da)b + a(db), \quad \forall a, b \in H$$

and such that $\Omega^1 = HdH$. This is a bit weaker than in usual differential geometry even when H is

commutative because we do not demand that 1-forms commute with elements of H . When H is a Hopf algebra one can ask that Ω^1 is translation-invariant, again in a certain sense involving the coproduct Δ .

In this sense a “quantum group” is not merely a Hopf algebra but has additional structure analogous to that of a Lie group. The flavour of this deeper theory is visible already in the geometry of the line and the circle, even though these are commutative as algebras. For example, one may classify all simple translation-invariant differential structures (Ω^1, d) (those with no proper quotient). For $\mathbb{C}[x]$ they turn out to be labelled by $\lambda \in \mathbb{C}$ and of the form

$$da(x) = \frac{a(x + \lambda) - a(x)}{\lambda} dx, \quad dx a(x) = a(x + \lambda) dx$$

of a finite difference on any $a \in \mathbb{C}[x]$. Only the case where $\lambda = 0$ has dx commuting with functions, so only this case is seen in usual geometry. For $\mathbb{C}[t, t^{-1}]$ the translation-invariant (Ω^1, d) are classified by $q \in \mathbb{C}^*$ and have the form

$$da(t) = \frac{a(t) - a(qt)}{(1 - q)t} dt, \quad dt a(t) = a(qt) dt$$

of a “ q -derivative” on any $a \in \mathbb{C}[t, t^{-1}]$. These two examples indeed reflect the two main types of quantum group known today. The most famous example here is the “ q -deformation” quantum group $\mathbb{C}_q[SL_2]$ with generators a, b, c, d , and with relations and coproduct

$$\begin{aligned} ba &= qab, \quad bc = cb \\ ca &= qac, \quad dc = qcd, \quad db = qbd \\ da &= ad + (q - q^{-1})bc, \quad ad - q^{-1}bc = 1 \end{aligned}$$

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where matrix multiplication is understood (so $\Delta a = a \otimes a + b \otimes c$, etc.). One has similarly $\mathbb{C}_q[G]$ versions for all compact Lie groups G and their complexifications. For an easy example of the other “ λ -deformation” type try $\mathbb{C}_\lambda[\mathbb{R}^{1,3}]$ defined as the algebra with generators t, x_i where $i = 1, 2, 3$; relations $[x_i, t] = i\lambda x_i$; and an additive coproduct as for $\mathbb{C}[x]$ above. This is actually the enveloping algebra of a solvable Lie algebra (see below) but viewed as a noncommutative coordinate algebra. Measurements from NASA’s GLAST satellite in 2007 may be able to test if our own spacetime could be like this, with $\lambda \sim 10^{-44}$ seconds if the effect comes from quantum gravity. More nontrivial examples of this type are the “bicrossproduct” quantum groups to be mentioned later.

Our second point of view on what a quantum group is starts with the observation that the group algebra kG of any group and the enveloping algebra $U(\mathfrak{g})$ of any Lie algebra again form Hopf algebras, this time with a symmetric coproduct (their coalgebra is “co-commutative”). The group algebra over k of any group

G is simply the vector space with basis G and product that of G on basis elements, extended linearly. We also have

$$\Delta x = x \otimes x, \quad \epsilon x = 1, \quad Sx = x^{-1}$$

for all $x \in G$, extended linearly. Similarly, let $(\mathfrak{g}, [,])$ be a Lie algebra with $[,]$ the Lie bracket. An easy (but not very elegant) way to define $U(\mathfrak{g})$ is to choose a basis of \mathfrak{g} and let $U(\mathfrak{g})$ be the free associative algebra with the basis elements as generators and relations $vw - wv = [v, w]$ for all basis elements v, w . Everything is extended linearly so this equation also holds for all $v, w \in \mathfrak{g}$. The coproduct is the additive one $\Delta v = v \otimes 1 + 1 \otimes v$ on the generators. In these examples an action of the algebra kG or $U(\mathfrak{g})$ is equivalent to a linear action of the underlying group or Lie algebra, while Δ encodes the rule for how actions extend to tensor products. Likewise a general Hopf algebra H can be viewed as a “generalised symmetry” where an element $h \in H$ acts by Δh in the tensor product. This in turn is needed, for example, to specify what it means for another algebra to be covariant under H .

The most famous example here is $U_q(sl_2)$ with generators e, f, q^h, q^{-h} (the abuse of notation is conventional) and

$$q^h e q^{-h} = q^2 e, \quad q^h f q^{-h} = q^{-2} f, \quad [e, f] = \frac{q^h - q^{-h}}{q - q^{-1}}$$

$$\Delta e = e \otimes q^h + 1 \otimes e$$

$$\Delta f = f \otimes 1 + q^{-h} \otimes f, \quad \Delta q^h = q^h \otimes q^h.$$

We require $q^2 \neq 1$. One has similarly $U_q(\mathfrak{g})$ for all \mathfrak{g} defined by a symmetrisable Cartan matrix. These quantum groups have a rich algebraic structure leading to knot and 3-manifold invariants. Among the deepest theorems is the existence of the Lusztig-Kashiwara canonical basis inducing bases for highest weight modules, which is remarkable even in the classical case when $q \rightarrow 1$.

A third point of view is that Hopf algebras are the next simplest category after Abelian groups admitting Fourier transform. This point of view is responsible for the large class of “bicrossproduct” quantum groups of self-dual form. They are simultaneously “coordinate” and “symmetry” algebras, and truly connected with quantum mechanics. An example is $\mathbb{C}[\mathbb{R}^3 \rtimes \mathbb{R}]_\lambda \blacktriangleright \blacktriangleleft U(so_{1,3})$ which is the Poincaré quantum group of the noncommutative spacetime algebra $\mathbb{C}_\lambda[\mathbb{R}^{1,3}]$ above. Here special relativity still applies but as a quantum group symmetry. At the same time this quantum group can be interpreted as the quantisation of a particle moving in a curved geometry with black-hole-like features.

Further reading

S. MAJID, *A Quantum Groups Primer*, L.M.S. Lect. Notes 292 (2002).