A System of Axioms of Set Theory for the Rationalists

Jan Mycielski

Introduction

This paper proposes and discusses a list of axioms for set theory based on the principle: *Accept as much regularity or specificity as possible without weakening the theory*.

The philosophy of mathematics has little or no influence upon 99% of mathematics. But there is that 1% where it matters, namely the choice of axioms of set theory, and this is the theme of this paper.

There are two extreme ontologies of mathematics: (a) Platonism, which tells us that *pure* mathematics is a description of an ideal structure that exists independently of humanity, and (b) Formalism, which says that *pure* mathematics is just a game with symbols. (Both views acknowledge the seminal role of applications, e.g., both agree that Greek geometry is an abstract approximate description of the physical space-time.) We think that neither (a) nor (b) is convincing; (a) assumes too much, it violates Ockham's principle *entia non sunt multiplicanda praeter necessitatem*, and (b) ignores that *logic and set theory constitute a framework and a tool for describing reality which was given to us by natural evolution*. We believe the latter since people of all cultures agree that mathematical arguments are convincing, and those who study the rules of logic and the axioms of set theory (ZFC with urelements allowed) think that they are evident. [Some postmodernists try to refute this observation by quoting various psychological experiments. We cannot take here the time and space to criticize them, but we believe that the evidence in favor of our opinion is overwhelming.] (Note: In this article, I will use parentheses to indicate additional information that is necessary but outside the main flow of ideas, square brackets for additions that are more remote from the main flow, and curly braces for digressions.)

Thus we accept a view which is intermediate between (a) and (b) and which says that not only applications but (in a great measure) human nature itself defines and causes pure mathematics. The ideal actually infinite sets of the Platonists are replaced by physical phenomena in human brains, that is, thoughts of things like boxes whose content is not fully imagined (see [H_1]). The meaning of quantifiers is explained as follows (see [SK] and [H_2]): If we claim in *pure* mathematics that $\forall x \exists y \varphi(x, y)$, where x and y range over a universe U, we assert only that we have a mental operation such that given any a in U we can imagine a b in U satisfying $\varphi(a, b)$. Hence the infinite sets and universes of pure mathematics are not actually but only potentially infinite. (For a fuller explanation see remark 3 at the end of this paper.) Thus pure mathematics is a finite human construction in a state of growth dealing with imaginary objects. It makes no sense to call it true or false since truth can appear only in applications (this does not contradict the fact that there exists a mathematical theory of the relation of truth). And yet logic and set theory are not arbitrary since human intelligence is made to describe reality in this framework (i.e., to classify using sets, sets of sets. etc.).

Jan Mycielski is professor emeritus in the Department of Mathematics at the University of Colorado in Boulder. His email address is `jmyciel@euclid.colorado.edu`.

{Although we have explained pure mathematics without introducing actual infinity, it seems that actual infinity does exist in physical reality, e.g., the space-time continuum appears to be infinite (see e.g., [P]). But some objects or structures of mathematics are purely *imaginary*, for example, a well-ordering of the real line, while others have potential interpretations as physical objects or processes, and we call them *real*. In mathematical practice many real objects are constructed or explained by means of imaginary ones. This is the natural way to do mathematics, such are the necessities of human intelligence. Constructivism, which tries to avoid imaginary objects, is unwieldy, but the distinction between imaginary and real objects is interesting, see [DM] and [M_5].}

Although the concept of truth does not apply to pure mathematics, we can ask *does such and such a set-theoretic proposition P constitute a natural law of thought*? Of course if the answer is *yes* we accept P as an axiom. If it is *no*, but P is consistent with the natural laws, then we are free to accept or to reject P. After Gödel and Cohen it is known that many simple set-theoretic propositions P are in that last category. And yet some of them can be desirable axioms if they have any of the following properties: (1) They simplify set theory, inducing regularities without excluding any interesting objects. (2) They strengthen set theory and enrich its universe with interesting objects.

For these reasons it is rational to add new axioms, when we think they satisfy (1) or (2). But I feel that, for a long time, set theorists have not taken advantage of this freedom; that is, they accept in practice the view of Platonists who worry that the prospective axioms could be false. (The only axioms extending ZFC which set theorists accept rather freely are the large cardinal axioms, see [Ka] and Axiom SC below.) For example, my paper [M_3] was written under the spell of that restrictive tradition. On the other hand, such new axioms cannot be written in stone. Since the future developments of mathematics may require their rejection, they can reflect only the actual state of mathematics.

The purpose of this paper is to propose and to discuss briefly a system ST of axioms for set theory which appear at present to be the natural choices of a rationalist. ST will be much stronger than the traditional theory ZFC, since several "conjectures" will be accepted as axioms. I will argue that the acceptance of these "conjectures" (they are known to be consistent with the original axioms if the latter are consistent) is well motivated.

The Axioms of ST

We propose a set theory ST based on ten axioms:

The Axiom of Extensionality:

(1) $$\forall xy[\forall z[z \in x \longleftrightarrow z \in y] \to x = y].$$

This axiom defines the concept of a set in terms of the membership relation $\in$. Since x and y are unrestricted variables, (1) also precludes the existence of objects that are not sets. This may appear too restrictive since in real life we imagine many objects which we do not treat as sets. Therefore in some older books the universe of set theory is divided into sets and non-sets (called *urelements*), and in the Axiom of Extensionality the range of x and y is restricted to sets; see e.g., [KM] and [Su]. However, experience has shown that in mathematics urelements are not essential (they can be constructed in terms of sets and a modification of the relation $\in$). Therefore, in view of its simplifying role, we accept the Axiom of Extensionality.

The Axiom of Union:

(2) $$\forall x \exists y \forall z[z \in y \longleftrightarrow \exists s[z \in s \,\&\, s \in x]].$$

Of course we often need to construct a set y in terms of a set x in the above way. Thus we accept the Axioms of Union. (We write $y = \bigcup(x)$.)

The Axiom of the Powerset:

(3) $$\forall x \exists y \forall z[z \in y \longleftrightarrow \forall s[z \in s \to s \in x]].$$

Once again, we often need to construct y in terms of x in the above way. Thus we accept the Axiom of the Powerset. (We write $y = P(x)$.)

The Axiom of Replacement:

(4) $$\begin{aligned}&\forall \bar{u}[\forall xy \exists z[\varphi(x,y,\bar{u}) \to y = z] \to \\ &\forall d \exists r \forall y[y \in r \longleftrightarrow \exists x[x \in d \,\&\, \varphi(x,y,\bar{u})]]].\end{aligned}$$

Here $\bar{u}$ denotes a finite string of variables, and φ is any formula written in terms of $\neg$ (negation), $\to$ (implication), $\forall$ and $\exists$ (universal and existential quantifiers), the symbols $=$ and $\in$, and variables, and such that the variables x, y, z, d, r do not appear in $\bar{u}$. This axiom is really a rule of proof since we can put for φ any formula we wish. It tells us that if we pick any string of sets $\bar{u}$, and a formula $\varphi(x, y, \bar{u})$ such that for all x there is at most one y which satisfies it, then for every set d (the domain) there exists a set r (the range) which is the image of d under φ. Again we often use this rule to construct r from d (and $\bar{u}$), and hence we accept the Axiom of Replacement.

[For example, if we choose φ to be the formula $x = y \,\&\, \psi(x, \bar{u})$, then r is the subset of d consisting of these x that satisfy $\psi(x, \bar{u})$. If we choose a φ that is always false then r is the empty set $\emptyset$. Then using the Powerset Axiom we can construct the set $d = PP(\emptyset) = \{\emptyset, \{\emptyset\}\}$. And, using this d and an appropriate $\varphi(x, y, a, b)$, (4) yields the unordered pair $\{a, b\}$. Then we can build the singleton $\{a\}$ and the ordered pair $\{\{a\}, \{a, b\}\}$, etc.]

The Axiom of Regularity:

(5) $$\forall x[x \neq \emptyset \to \exists y[y \in x \,\&\, \forall s[s \in y \to s \notin x]]].$$

The only role of this axiom is to simplify the universe of sets. It precludes the existence of infinite sets $\{a_1, a_2, \dots\}$ such that $a_1 \ni a_2 \ni a_3, \dots$. Indeed if x was such a set it would violate (5). It also precludes sets a such that $a \in a$. Indeed, for such an a, the set $x = \{a\}$ would violate (5). Of course any urelements would also violate (5). Set theories without the axiom of regularity have been considered, but they do not appear to lead to any sufficiently interesting mathematics. Therefore, in view of its simplifying role, we accept the Axiom of Regularity.

However, we will introduce below an axiom (7) which implies (5); thus (5) is superfluous in ST, but it will appear in some later remarks.

The Axiom of Infinity:

(6) $\exists x[x \neq \varnothing \;\&\; \forall y[y \in x \to y \cup \{y\} \in x]]$,

where $y \cup \{y\} = \bigcup\{y, \{y\}\}$. This axiom is essential for the construction of infinite sets, for example, of the set $\mathbb{N}$ of positive integers. The former axioms (1)–(5) of set theory constitute a system definitionally equivalent to Peano's Arithmetic (PA), and this system is not strong enough to develop mathematics in a natural way. For example, (6) is necessary for the development of analysis.

[A very artificial finitistic way of doing set theory is possible. It is based on the Completeness Theorem of Gödel. Namely, we can develop mathematics within the theory PA + Con(S), where PA is Peano's Arithmetic and Con(S) expresses in the language of PA (by means of Gödel numbers) the consistency of a set theory S. In this theory we can define a model of S. But this is not natural since it is only a translation of the idea of S into the language of PA.]

The Axiom V = OD:

From now on we depart from the beaten track since V = OD and the remaining axioms have not yet been accepted by other set theorists. To explain this axiom, recall first that the class of ordinal numbers Ord is defined to be the smallest class of sets that contains $\varnothing$ and that is closed under unions of its subsets and closed under the function $x \mapsto x \cup \{x\}$. (One shows that $\alpha \in Ord$ if and only if $\forall xy[x \subseteq y \subseteq \alpha \to x \in y \in \alpha]$.) The former axioms (1)–(6) yield a proof that each ordinal α is well-ordered by the relation $\in$. As usual, $\in$ restricted to ordinals is denoted by $<$, and ω denotes the least infinite ordinal. For every ordinal α, we define $\alpha + 1 = \alpha \cup \{\alpha\}$. Then we define recursively the sets V_α ($\alpha \in Ord$):

$$V_\alpha = \bigcup_{\xi<\alpha} P(V_\xi).$$

Thus $V_0 = \varnothing$, $V_1 = \{\varnothing\}$, $V_2 = \{\varnothing, \{\varnothing\}\}, \dots, V_{\alpha+1} = P(V_\alpha), \dots$. The former axioms (1)–(6) yield the theorem:

$$\forall x \exists \alpha [x \in V_\alpha],$$

and we write

$$(*) \qquad V = \bigcup_{\alpha \in Ord} V_\alpha.$$

Thus V denotes the universe of all sets. Unlike the V_α's, V is not a set, and hence $(*)$ is not a formal definition in the language of set theory.

Now we form the models $\langle V_\alpha, \in\rangle$, and we denote by D_α the set of elements of V_α which can be defined by unary formulas in the model $\langle V_\alpha, \in\rangle$. Then OD is a class informally defined as follows:

$$\text{OD} = \bigcup_{\alpha \in Ord} D_\alpha.$$

Again OD is not a set. But our seventh axiom, V = OD, can be formally expressed as follows:

(7) $\forall x \exists \alpha [x \in D_\alpha]$.

Notice that each D_α is finite or countable. Still the union $\bigcup_{\xi<\alpha} D_\alpha$ builds up relentlessly, so we never need in mathematics any set that has to be outside of OD. [Of course we could assume that there exist such sets, but heretofore this assumption has not led to any interesting mathematics.]

We mention three consequences of V = OD: (a) it implies the Axiom of Choice, and moreover it yields a certain binary formula $\varphi(x, y)$ that well orders all of V; (b) it implies the Axiom of Regularity (5); (c) the set theory S based on the axioms (1)–(7) has the elegant property that the definable elements of any model M of S constitute an elementary submodel of M. Peano's arithmetic also has this property, but the traditional system of axioms ZFC does not have it. I believe that, in view of these consequences and for sake of definiteness, it is rational to accept V = OD.

Of course this may be a temporary situation. For example, some interesting theory involving real numbers that are not in OD could arise in the future. But we have no reason to predict that such a thing will happen.

[It appears natural to add a refinement (7*) of (7), which, in the presence of (5), implies (7):

$$(7^*) \qquad V_\alpha \subseteq \bigcup_{\xi<|V_\alpha|} D_\xi,$$

where $|V_\alpha|$ is the ordinal of the least well-ordering of V_α. But I do not know any interesting consequences of (7*). Every set x has the structure of a tree $\langle Tr(x), \in\rangle$, where

$$Tr(x) = \{x\} \cup x \cup \bigcup(x) \cup \bigcup(\bigcup(x)) \cup \dots$$

Perhaps one can postulate some more detailed relation between the definitions of definable sets and their trees?]

The Axiom GCH:

The cardinal number of a set a, in symbols $|a|$, is the smallest ordinal number which has a bijection to a. Thus the least infinite cardinal number is ω, also denoted $\aleph_0$. The next one is denoted ω_1 or $\aleph_1$, etc. For every cardinal number α we define $2^\alpha = |P(\alpha)|$ and α^+ = (the least cardinal larger than α). The Axiom GCH is:

$$\text{(8)} \qquad \textit{For every infinite cardinal } \alpha \textit{ we have } 2^\alpha = \alpha^+.$$

This axiom greatly simplifies the theory of infinite cardinal numbers, and it adds many interesting theorems to the combinatorics of infinite sets. These well known advantages are so significant that it is rational to accept GCH as an axiom of set theory. (Even CH, that is $2^{\aleph_0} = \aleph_1$, has many interesting consequences.)

Set theorists often say that probably GCH restricts too much the sets $PP(\alpha)$. But one can also surmise the opposite. Indeed $2^\alpha > \alpha^+$ precludes the existence of any subset of $P(\alpha)$ which codes a function $f : P(\alpha) \to P(\alpha)$ such that whenever $x, y \in P(\alpha)$ and $x \neq y$, then $f(x)$ and $f(y)$ code different well-orderings of α. Since, as we explained in the first section, $PP(\alpha)$ is only potentially infinite, we are free to accept GCH. [It is often said that the Axiom of Choice (AC) and CH have consequences that contradict probabilistic intuition that is based on physical experience. However, a closer look shows that those paradoxical consequences do not pertain to any mathematical objects that have a potential for direct physical interpretations (for a detailed discussion see [DM] and [M_5]). On the other hand AC and GCH have similar organizing or simplifying roles, which motivate their presence in ST. (As mentioned earlier AC is a consequence of V = OD.)]

The acceptance of GCH leads us to the following considerations. If we have a nontrivial proof of a theorem T which does not use GCH, such that T becomes trivial if GCH is assumed, then that proof ought to give a stronger theorem T* that is still nontrivial even in the presence of GCH. I will give two examples where I do not know the correct statement of T*.

The first is a theorem of R. McKenzie and S. Shelah [MS]. To state it we need the following concepts. An algebra $\mathbf{A}$ of countable type is a system $\langle A, f_1, f_2, \dots \rangle$, where A is a nonempty set and each f_n is a function of finitely many variables running over A and with values in A. Let Σ be an infinite system of equations written in terms of the f_n's and any (possibly infinite) number of unknowns. $\mathbf{A}$ is said to be *equationally compact* if every Σ has the property that if all its finite subsystems can be solved in $\mathbf{A}$ then the entire system Σ can be also solved in $\mathbf{A}$. And, $\mathbf{A}$ will be called *folded* if for every proper homomorphic image $\mathbf{B}$ of $\mathbf{A}$ there exists a finite system Σ which can be solved in $\mathbf{B}$ but not in $\mathbf{A}$. It was known (W. Taylor [T]), that if $\mathbf{A}$ is of countable type, equationally compact, and folded then $|A| \leq 2^{\aleph_0}$. McKenzie and Shelah proved without using CH that $\aleph_0 < |A| < 2^{\aleph_0}$ is impossible. According to the idea expressed earlier, the proof should yield a stronger theorem T* which remains nontrivial even if we assume the theorem of Taylor and CH. I do not know such a theorem.

Another example of this situation is the following. A well known conjecture of R. L. Vaught says that if T is a countable theory, then the number α of isomorphism types of countable models of T cannot satisfy $\aleph_0 < \alpha < 2^{\aleph_0}$. Morley [Mo] has shown a little less, namely that $\aleph_1 < \alpha < 2^{\aleph_0}$ is impossible. Again I think that a stronger conjucture and a theorem that do not follow immediately from CH should exist.

The above ideas should not be construed as a criticism of a branch of foundations called Reverse Mathematics. In this branch one proves theorems of the form T → A, where T is some interesting theorem and A is an axiom (of course A is not assumed in the proof of T → A). Some examples of such theorems are the following. *Tarski's theorem*: (For all infinite sets X there exists a bijection of X to $X \times X$) → (Axiom of Choice). Or *Sierpiński's theorem*: (The space $\mathbb{R}^3$ with a Cartesian coordinate system X, Y, Z, is a union of three sets A, B, and C such that every linear section of A parallel to X is finite, every linear section of B parallel to Y is finite, and every linear section of C parallel to Z is finite) → CH. There are many interesting theorems of Reverse Mathematics, but some critics do not care for such results. [Tarski told me the following story. He tried to publish his theorem (stated above) in the *Comptes Rendus Acad. Sci. Paris* but Fréchet and Lebesgue refused to present it. Fréchet wrote that an implication between two well known propositions is not a new result. Lebesgue wrote that an implication between two false propositions is of no interest. And Tarski said that after this misadventure he never tried to publish in the *Comptes Rendus*.]

The Axiom SH:

(9) *If A is a linearly ordered set such that every set of disjoint open intervals of A is countable then A has a countable subset which intersects every non-empty open interval of A.*

This axiom, called Suslin's Hypothesis, has been extensively studied (see [Ku]). Once again, we do not meet in mathematics any linear orders violating (9). So we accept (9) since it simplifies set theory in a natural way.

It may be of some interest to recall a statement equivalent to (9) (see e.g., [Ku]). By a tree we mean a partially ordered set T such that the set of predecessors of any element of T is fully well-ordered. A subset of T is called a chain if and only if it is well-ordered; it is called an antichain if no two of its elements are comparable. Then (9) can be expressed equivalently as follows:

(9′) *If every chain and every antichain of a tree T is countable then T is countable.*

(Perhaps the simplifying nature of (9′) is more salient than that of (9). SH or (9′) may suggest similar axioms for higher cardinal numbers.)

The Axiom $AD^{L(\mathbb{R})}$:

To explain this axiom we need the following concepts. For every set A, we form the relational structure $\langle A, \in\rangle$, where $\in$ is restricted to A. Then a set $X \subseteq A$ is called *A-constructible* if there exists a formula of set theory $\varphi(x, \bar{y})$ and a finite string $\bar{a}$ of elements of A such that

$$x \in X \longleftrightarrow (\varphi(x, \bar{a}) \text{ is true in } \langle A, \in\rangle).$$

Let $C(A)$ denote the set of A-constructible subsets of A.

Then we define

$$L_\alpha = \bigcup_{\xi<\alpha} C(L_\xi)$$

and

$$L = \bigcup_{\alpha\in Ord} L_\alpha.$$

We define also

$$L_0(\mathbb{R}) = V_{\omega+1},$$

and, for all $\alpha > 0$,

$$L_\alpha(\mathbb{R}) = \bigcup_{\xi<\alpha} C(L_\xi(\mathbb{R})),$$

and finally

$$L(\mathbb{R}) = \bigcup_{\alpha\in Ord} L_\alpha(\mathbb{R}).$$

(The notation $L(\mathbb{R})$ derives from the existence of natural bijections from $V_{\omega+1}$ to the set $\mathbb{R}$ of real numbers.) The structures $\langle L, \in\rangle$ and $\langle L(\mathbb{R}), \in\rangle$ are of special interest. The first satisfies all the axioms (1)–(8) (but not (9)), and the second satisfies (1)–(6). In fact $\langle L(\mathbb{R}), \in\rangle$ is the smallest structure which contains $\mathbb{R}$ and all the ordinal numbers and which satisfies (1)–(6).

Although $L(\mathbb{R})$ is minimal in the above sense it is large enough for mathematical analysis. For example, it contains not only all the real numbers but also the projective sets of all ranks $< \omega_1$, and presumably all sets that are of true significance for analysis over Polish spaces. On the other hand, it does not contain sets that appear pathological in a probabilistic sense. But these claims depend on the axiom $AD^{L(\mathbb{R})}$ which we will explain presently.

Consider the following infinite binary game of perfect information. Let $\{0,1\}^\omega$ be the set of all infinite sequences $(\varepsilon_0, \varepsilon_1, \ldots)$ where $\varepsilon_n \in \{0,1\}$, and let a set $X \subseteq \{0,1\}^\omega$ be given. Player I chooses ε_0, then player II chooses ε_1, then again I chooses ε_2, and II chooses ε_3, etc. The set X and the sequence $(\varepsilon_0, \ldots, \varepsilon_{n-1})$ are known to the player choosing ε_n. I wins if the sequence $(\varepsilon_0, \varepsilon_1, \ldots)$ belongs to X and II wins otherwise.

The Axiom of Determinacy AD is the statement *for every X one of the players has a winning strategy*. It is easy to prove using the Axiom of Choice that AD is false. But the Axiom $AD^{L(\mathbb{R})}$ [which was suggested in [M_6S] and in [M_4] footnote ([1])] is the following restriction of AD:

(10) AD *is true provided $X \in L(\mathbb{R})$.*

This axiom has many interesting consequences. Assuming $AD^{L(\mathbb{R})}$ the class $L(\mathbb{R})$ becomes the natural universe of sets for mathematical analysis in Polish spaces. Indeed, AD implies that: *all uncountable sets of reals have perfect subsets, all sets of reals are Lebesgue-measurable, and all have the property of Baire* (see [M_2]). Also the theory of projective sets gets a very regular form (see e.g., [M]).

Therefore it is rational to accept the axiom $AD^{L(\mathbb{R})}$.

The Axiom SC:

To explain this axiom we need the following concepts. For every infinite cardinal α, a Hausdorff space S is called *α-compact* if every covering of S with open sets has a subcovering with less than α sets. (Thus ω-compact means compact in the usual sense.) A cardinal α is called *strongly compact* if every topological Cartesian product of any number of α-compact spaces is α-compact. By the Tychonoff product theorem, ω is a strongly compact cardinal. (There exist other definitions of strongly compact cardinals. They were introduced in [KT] and the above definition was shown in [M_1].) The axiom SC is the following:

(11) *For every cardinal κ there exists a strongly compact cardinal larger than κ.*

It is natural to replace the product topology in the definition of a strongly compact cardinal α by a larger topology whose basis is the set of all cylinders over products of less than α open sets. But the corresponding concept of strong compactness is equivalent to the former.

Thus SC postulates the existence of many cardinal numbers similar to ω. One can prove many large cardinal properties of α-compact cardinals, for example they are strongly inaccessible and even measurable (see [D] and [Ka]).

The axiom SC is also interesting for other reasons. One of them is a theorem of R. M. Solovay [So], which says that all cardinals α, which are larger than the least uncountable strongly compact cardinal and are singular and strong limit[1], satisfy $2^\alpha = \alpha^+$.

[1] *α is strong limit if $\kappa < \alpha \to 2^\kappa < \alpha$.*

{Again we believe that the proof in [So] should yield a property of α stronger than $2^\alpha = \alpha^+$, which does not become obvious under the assumption of GCH.}

To state an interesting consequence of SC let us generalize the infinite game defined in the previous section. We replace the set $\{0, 1\}$ by an arbitrary set P, and the set X by any $X \subseteq P^\omega$. (Thus the players I and II choose their ε_n in P.) Let N be a countable set, and consider the product topology in $P^\omega \times N^\omega$, where both P and N are given the discrete topology. A set $X \subseteq P^\omega$ is called *analytic* if it is a projection of a closed subset of $P^\omega \times N^\omega$. It is a consequence of SC that *if X is analytic then the game is determined, i.e., one of the players has a winning strategy.* (In fact a large cardinal axiom significantly weaker than SC suffices to prove this theorem, viz. $(\exists\kappa > |P|)\ [\kappa \to (\omega_1)_2^{<\omega}]$, see [M$_2$]. This result for $P = \omega$ is due to D. A. Martin; in [M$_2$] his proof is generalized to all sets P.)

Large cardinal axioms much stronger than SC have been proposed and studied. Some of them imply the axiom $\mathrm{AD}^{L(\mathbb{R})}$ (this is a difficult theorem of Martin, Steel, and Woodin, see [N$_1$], [N$_2$] and [Ka]), but I stated SC rather than those stronger axioms since the latter are more complicated and, as far as I know, unlike SC, they are not suggested by any properties of ω.

Conclusion

This concludes my definition of a set theory ST which I believe to be reasonable, that is, as strong and simple as possible and unrestricted by any Platonic beliefs. Thus

$$\mathrm{ST} = [\mathrm{ZF} + (\mathrm{V} = \mathrm{OD}) + \mathrm{GCH} + \mathrm{SH} + \mathrm{AD}^{L(\mathbb{R})} + \mathrm{SC}],$$

where, as usual, ZF denotes the system (1)–(6). But, as explained in the introduction, ST is an attempt at a good synthesis of the *current* state of mathematics. It will have to be strengthened or modified if mathematics calls for more sets.

However, much of the current work in set theory consists of difficult and ingenious proofs in theories weaker than ST (see e.g., [Ka], [KL], and [S]), and of constructions of very artificial models that yield independence and consistency results. Of course this is interesting to the specialists, but I think that it is difficult to justify such work to mathematicians at large. Indeed they can object: We are not very interested in methodology; if you have the freedom to assume strong and simplifying axioms why don't you assume them?

Recently W. H. Woodin and others have proposed set theories that are inconsistent with ST, but I think that the motivation of ST is better (see remark **2** below).

[It is known that in very strong set theories, e.g., ZFC + (there exists a supercompact cardinal), one can prove that ST is consistent. But the definitions of supercompact cardinals or any cardinals sufficient for that proof (see [N$_1$, N$_2$]), are so complicated that the claim that ST is consistent is more convincing to me than the claim that these very strong theories are consistent.]

Additional Remarks

Let ZFC denote (as usual) the system of axioms (1)–(6) plus the Axiom of Choice. Let me reiterate the motivation of ST. As we mentioned in the Introduction, ZFC is natural in the sense that almost every mathematician who reads its axioms feels that he accepts them. However, as explained in our discussion of axioms (1) and (5), ZFC departs from the natural way of thinking by accepting some simplifications which eliminate certain sets that are not important for mathematics (urelements and sets that are not well founded). So it is natural to follow this path and accept the other axioms of ST that simplify the theory, namely $V = \mathrm{OD}$, GCH, SH, and $\mathrm{AD}^{L(\mathbb{R})}$. (Of course SC enriches rather than simplifies.)

This suggests the question why these well known propositions are not yet generally accepted by most set theorists. I see three reasons: (a) the tradition of treating them as open problems; (b) the thought that they oversimplify set theory; (c) the belief of Platonists that they could be false. In the next three sections I will argue contra (a), (b), and (c).

1. *Ad* (*a*). Of course (a) should be dismissed since it is known that none of the axioms (7)–(11) is a consequence of the other ones.

2. *Ad* (*b*). If we agree that ST does not appear to impose any bounds on the consistency strength of its possible extensions, then the fear that it oversimplifies set theory has no motivation. Thus I feel that (b) is not true (at least at the present time).

However, alternative theories were proposed recently in [W$_1$, W$_2$]. These theories yield certain descriptions of the model

$$\langle P(\omega_1), \omega_1, +, \cdot, \in\rangle,$$

where + and $\cdot$ are ordinal addition and multiplication restricted to countable ordinals, and they happen to disprove the Continuum Hypothesis; they prove $2^{\aleph_0} = \aleph_2$. This looks odd, and it is a big complication of the theory of cardinal numbers or of the combinatorics of infinite sets. Moreover, all uncountable subsets of ω_1 (and of $\mathbb{R}$) are imaginary objects without the potential for any direct physical interpretations (see [DM] and [M$_5$]). Hence any additions to ZFC describing these objects can be motivated only by human preference. Therefore the only objective criteria which can guide our choice among these theories are precisely the simplicity of the axioms and the regularity of their consequences. Are the theories proposed in [W$_1$, W$_2$] so attractive from this point of view that we should give up GCH?

[Some philosophers have tried to dismiss the concept of simplicity of a theory, claiming that it is vague or language-dependent or irrelevant. Yet the simpler theories are easier to communicate and easier to to remember, and in our descriptions of reality (that appear to be true) the simplest are the most convincing. Moreover, all generalizations or inductive inferences can be viewed as simplifications of lists of special cases. Therefore it is natural to apply also the criterion of simplicity or elegance in our choice of set-theoretic axioms and their consequences.]

3. *Ad* (*c*). Let me amplify some remarks made in the Introduction. Hilbert's view [H_1] of the structure of sets of pure mathematics as a finite array of potentially infinite sets can be compared to the interpretation of complex numbers as points of the Cartesian plane (by Wessel, Argand, and Gauss). Like the latter it gives a physical significance to some formal concepts. I think that the idea of Hilbert is deep since it simplifies in a dramatic way the ontology of pure mathematics. [It may have been anticipated by Poincaré, by Skolem (in some papers related to [Sk]), and even by Aristotle.] And yet this idea is not yet a part of the general mathematical culture (perhaps because it has little relevance outside of set theory or because of a weakness of the current philosophical culture). Now, a full understanding of this interpretation also requires an explanation of quantifiers that does not use actual infinity. None of the books that I know presents this development in modern terms, although this is very easy:

Let $\bar{x}$ and $\bar{y}$ be finite strings of variables, and $|\bar{s}|$ denotes the length of the string $\bar{s}$. Let ε be an operator which attaches to every formula $\varphi(\bar{x}, \bar{y})$ *without quantifiers*, where $\bar{x}$ and $\bar{y}$ are disjoint strings, a string of $|\bar{y}|$ new function symbols of $|\bar{x}|$ variables each. Denoting by $\varepsilon_{\varphi,\bar{y}}$ this string of new function symbols (if $\bar{x}$ is of length 0 they are constants) we have the axiom

$$\text{(H)} \qquad \varphi(\bar{x}, \bar{y}) \to \varphi(\bar{x}, \varepsilon_{\varphi,\bar{y}}(\bar{x}))$$

essentially due to Hilbert [H_2]. Granted this axiom, quantifiers can be defined as abbreviations

$$\exists \bar{y} \varphi(\bar{x}, \bar{y}) = \varphi(\bar{x}, \varepsilon_{\varphi,\bar{y}}(\bar{x}))$$

and

$$\forall \bar{y} \varphi(\bar{x}, \bar{y}) = \varphi(\bar{x}, \varepsilon_{\neg\varphi,\bar{y}}(\bar{x})).$$

Then the usual rules of logic concerning quantifiers can be derived from (H). Also using these formulas and working from inside out, variables (and quantifiers) can be eliminated from every sentence; and (H) can be viewed as an axiom-schema or a rule, where $\bar{x}$ and $\bar{y}$ are arbitrary strings of names of constants.

{In the presence of $V = \mathrm{OD}$ we have a definable well-ordering of the universe, and then the operator ε can be also defined: $\varepsilon_{\varphi,\bar{y}}(\bar{x})$ is the least $|\bar{y}|$-tuple such that $\varphi(\bar{x}, \varepsilon_{\varphi,\bar{y}}(\bar{x}))$ holds, and, if no such $|\bar{y}|$-tuple exists, then $\varepsilon_{\varphi,\bar{y}}(\bar{x})$ can be any $\bar{y}$-tuple, say $(\emptyset, \dots, \emptyset)$.

Logicians who want to interpret symbols in models (within set theory) can interpret the sequence $\varepsilon_{\varphi,\bar{y}}$ as a variable $|\bar{y}|$-tuple ranging over the relation (depending on $\bar{x}$) denoted by φ when the latter is nonempty, and unrestricted when it is empty.}

We conclude that the feeling of concreteness and reproducibility of mathematical objects is based on the fact that, no matter what language we use to describe them, they constitute finite structures in our thoughts and memories of very definite kinds. And the feeling of consistency of ZFC arises from the simplicity of these constructions. Thus we are able to explain these feelings without the assumption that mathematics describes some Platonic ideas independent of humankind.

References

[D] F. R. Drake, *Set Theory, An Introduction to Large Cardinals*, North–Holland, 1974.

[DM] R. Dougherty and J. Mycielski, Canonical universes and intuitions about probabilities, (to appear).

[H_1] D. Hilbert, On the foundations of logic and arithmetic (1904), *From Frege to Gödel, A Source Book in Mathematical Logic, 1879–1931*, (J. van Heijenoort, ed.), Harvard Univ. Press, Cambridge, 1976, pp. 129–138.

[H_2] ______, Die Logischen Grundlagen der Mathematik, *Math. Annalen* **88** (1923), 191–165.

[Ka] A. Kanamori, *The Higher Infinite*, Springer–Verlag, Berlin, Heidelberg, 2003 (second edition).

[Ku] K. Kunen, *Set Theory, An Introduction to Independence Proofs*, North–Holland, 1992.

[KL] V. G. Kanovei and V. A. Lyubetskii, On some classical problems of descriptive set theory, *Russian Math. Surveys* **48**:5 (2003), 839–927.

[KM] K. Kuratowski and A. Mostowski, *Set Theory*, North–Holland, 1968.

[KT] H. J. Keisler and A. Tarski, From accessible to inaccessible cardinals, *Fund. Math.* **53** (1964), 225–308.

[M] D. A. Martin, Descriptive set theory, *Mathematical Logic* (J. Barwise, ed.), North–Holland, 1991, pp. 783–815.

[MS] R. McKenzie and S. Shelah, The cardinals of simple models for universal theories, *Proc. Symp. Pure Math., vol XXV, Proceedings of the Tarski Symposium*, Amer. Math. Soc., Providence, RI, 1974, pp. 53–74.

[Mo] M. Morley, The number of countable models, *J. Symbolic Logic* **35** (1970), 14–18.

[M_1] J. Mycielski, Two remarks on Tychonoff's Product Theorem, *Bull. Acad. Polon. Sci.*, *Series Math. Astr. Phys.* **12** (1964), 439–441.

[M_2] ______, Games with perfect information, *Handbook of Game Theory with Economic Applications* (R. J. Aumann and S. Hart, eds.), North–Holland, 1992, pp. 41–69.

[M_3] ______, Axioms which imply GCH, *Fund. Math.* **176** (2003), 193–207.

[M_4] ______, On the axiom of determinateness, *Fund. Math.* **53** (1964), 205–224.

[M5] ______, Pure mathematics and physical reality, to appear in *Fundamental and Applied Mathematics* (*Fundamentalnaya i prikladnaya matematika*) **11**(5), 2005.

[M6S] J. Mycielski and H. Steinhaus, A mathematical axiom contradicting the axiom of choice, *Bull. Acad. Polon. Sci., Series Math. Astr. Phys.* **10** (1962), 1–3.

[N1] I. Neeman, Optimal proofs of determinacy, *Bull. Symbolic Logic* **1** (1995), 327–339.

[N2] ______, Optimal proofs of determinacy (II), *J. Math. Logic* **2** (2002), 227–258.

[P] R. Penrose, *The Road to Reality*, A. A. Knopf, New York, 2005.

[S] S. Shelah, Logical dreams, *Bull. Amer. Math. Soc.* **40** (2003), 203–228.

[Sk] T. Skolem, Logico-combinatorial investigations in the satisfiability or provability of mathematical propositions (1920), in *From Frege to Gödel, A Source Book in Mathematical Logic, 1879–1931*, (J. van Heijenoort, ed.), Harvard Univ. Press, Cambridge, 1976, pp. 252–263.

[So] R. M. Solovay, Strongly compact cardinals and the GCH, *Proc. Symp. Pure Math., vol XXV, Proceedings of the Tarski Symposium*, Amer. Math. Soc., Providence, RI, 1974, pp. 365–372.

[Su] P. Suppes, *Axiomatic Set Theory*, Dover Publ. Inc., New York, 1972.

[T] W. Taylor, Some constructions of compact algebras, *Annals of Math. Logic* **3** (1971), 395–436.

[W1] W. H. Woodin, The Continuum Hypothesis, *Notices Amer. Math. Soc.* **48** (2001), Part I, 567–576; Part II, 681–690; correction to Part II, ibidem **49** (2002), 46.

[W2] W. H. Woodin, Set theory after Russell: The journey back to Eden, *One Hundred Years of Russell's Paradox*, (G. Link, ed.), de Gruyter, Amsterdam, New York, 2004, pp. 29–47.

About the Cover

ICM Madrid 2006

As Manuel de León and Allyn Jackson explain elsewhere in this issue, the next International Congress of Mathematicians will be held in the summer of 2006 in Madrid. As many mathematicians already know, a number of extremely handsome posters have been distributed to advertise the event. The image on this issue's cover, which shows the cupola of the *Sala de las dos Hermanas* in the Alhambra, is taken from one of them. Two of the posters are shown in the article by Allyn Jackson, and the other two are reproduced below. The verses by Ibn Zamrak, mentioned in a caption in Jackson's article, are just visible on the cover. One of the posters below exhibits a view of the Colegio de las Teresianas, designed by the Barcelonian architect Gaudí, and the other the cupola of the imperial Escorial Palace just outside Madrid. The graphics designer for all of the posters associated with the ICM 2006 was Maria Casassas of Barcelona. The photographer was Marc Llimargas, who specializes in architectural photography. In particular, he did the photography for a recent book on Gaudí.

The geometric nature of Islamic design, incorporating complex symmetries, has been well-explored from a mathematical point of view. A fairly sophisticated discussion, referring specifically to the Alhambra, can be found in the book *Classical Tessellations and Three-manifolds* by José María Montesinos. One good introduction to the Alhambra, with a short discussion of the mathematics in context, is the book *The Alhambra* by Oleg Grabar. A mathematical treatise much respected by nonmathematicians is the University of Zürich Ph.D. thesis of Edith Müller, *Gruppentheoretische und Strukturanalytische Untersuchungen der Maurischen Ornamente aus der Alhambra in Granada.*

Our thanks to Manuel de León for his help in obtaining the images we used.

—*Bill Casselman, Graphics Editor*
(notices-covers@ams.org)