Math Circles and Olympiads

MSRI Asks: Is the U.S. Coming of Age?

James Tanton

The list of outreach programs aimed at providing rich mathematical experiences for middle- and high-school students is growing. Programs such as the Berkeley Math Circle [6], the San Jose Math Circle [8], and the Boston Math Circle [7] are thriving; a plethora of summer math camps exist across the nation; and student participation in regional, national, and international mathematics competitions is significant. Clearly there is some important issue being addressed by these programs. It's exciting and intriguing, even if the "it" cannot be easily articulated.

On December 16–18, 2004, the Mathematical Sciences Research Institute (MSRI) took the bold step to bring together over one hundred dedicated folk, all with strong interests in these programs and all clearly committed to the larger goal of sharing the joy of pure mathematics. Organized by Hugo Rossi, Deputy Director of MSRI; Tatiana Shubin of San Jose State University, CA; Zvezdelina Stankova of Mills College, CA; and Paul Zeitz of the University of San Francisco, the *Conference on Math Circles and Olympiads* united educators and researchers from the pre-college and college worlds, brought focus to the questions of "what are we doing?" and "where are we going?," and offered concrete steps towards fostering discussion and sharing resources. From it, MSRI plans to establish a permanent national educator/researcher network. "One of the central purposes of this conference was to bring these communities together to begin an interaction," writes Hugo Rossi. And it seems that the math circle concept provided a key intersection point of discussion. What route—alternative to math competitions and math camps—do circles provide for the discovery of young talent? Is the use of the word "talent" appropriate? Can circles contribute to the general secondary curriculum? Are math circles self-sustaining? What makes them work? Hugo Rossi comments:

> I am convinced that the idea of math circles has come of age in the U.S., and it can become a movement which develops fast so as to be something like what exists in Eastern Europe. The people at our conference are the resource for this development; we have to make that resource widely available.

MSRI recognizes that the time is right to draw upon the collective experiences of our colleagues—here and abroad—and examine the opportunities that lie before us. I am personally intrigued by the broader challenge of possibly incorporating math circle ideals into the fixed secondary curriculum. (I have had the pleasure of working with Bob and Ellen Kaplan of the Boston Math Circle for a number of years before leaving the college world to tackle life as a high-school teacher.) I have no real answers, but I was delighted to learn from this conference that I am far from alone in exploring issues like these. Serious discussion about the "it" that math circles and other extracurricular activities provide is now under way.

The Math Circle Experience

Extracurricular circles in a variety of subjects began in Hungary in the 1800s, all with the goal of providing young students opportunities to pursue personal interests to the fullest. Today they are considered a standard part of the Eastern-European

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student experience, and participation in them is regarded as just as natural as participation in sports activities is viewed in the U.S. Although there is no set protocol to a math circle experience, all circles have the same goal of sharing the intellectual appeal and beauty of mathematics with as large an audience as possible. They engage faculty from both secondary and post-secondary institutions in their operation and successfully welcome students of all backgrounds to the mathematical experience. Circles now exist in many countries, including the U.S. (see also [1], for instance), and follow multiple styles and approaches. Given the success of the Eastern European model it is natural to ask then whether some version(s) of the math circle experience could be incorporated into the U.S. cultural norm. Could even more be accomplished? Tatiana Shubin notes:

> We're in a wonderful and unique situation … where we have the widest source of practices and traditions from all over the world to draw upon. And there exist new tools, like the Internet and TV … If real circle meetings could be aired on TV, lots of people would see how kids interact with it—and it might make a profound impact on the public's perception of our beloved discipline.

Indeed, imagine the impact! As a beginning step, participants at the conference were treated to two demonstration classes—one from each of the Boston and Berkeley Math Circle programs— and it was clear each time that indeed something remarkable was taking place.

Two U.S. Models

The Boston Math Circle was founded by Bob and Ellen Kaplan and Tomas Guillermo in 1994 and currently has over 120 participants. In this circle, the lecture format is completely set aside and mathematics is discovered and developed through exploration, intellectual play, and the give-and-take of conversation (see [4]). The questions discussed are attractive and mathematically rich and offer multiple pathways for exploration, generalization, and variation. Students work on the same fundamental question and the ideas generated from it for ten consecutive weeks. As examples, young students, K–3, have explored the vague question "Are there numbers between numbers?" to discover, by the end of the semester, the density and the countability of the rationals. Middle schoolers, in exploring the issue of whether or not a power of two ever begins with a seven, created their own versions of logarithms, developed basic results in ergodic theory, and proved density results on infinite sets. Slightly older students have found their own means to compute i^i, to prove the fundamental theorem of algebra, and to conduct original research [5]. The Boston Math Circle works hard to remove any sense of competition and completely disregards labels of "talented" and "gifted". It relies solely on the "intellectual seduction of attractive questions," as Ellen puts it, to engage and excite. The role of an instructor is not to teach, but to guide, nudge, offer suggestions, and, more often than not, to step out of the way.

The Berkeley Math Circle, founded in 1998 and run by Zvezdelina Stankova, works with over 50 San Francisco Bay Area middle- and high-school students. It openly recognizes that there are many different routes for the enjoyment of mathematics and actively works to offer a variety of experiences. Meetings tend to vary in style, organization, and topic from week to week, and competition and competition preparation play an important role in the circle experience. (The Berkeley circle has had tremendous success helping students prepare for national and international competitions.) Stankova also recognizes that great joy and beauty can be found in advanced mathematics and may preface a session with a lecture on a sophisticated topic. For example, the following is a Berkeley Math Circle favorite:

> Four planar circles are pair-wise externally tangent. Three of the circles are also tangent to a line L. If the radius of the fourth circle is one unit, what is the distance of its center from L?

Participants tackle this problem after attending a lecture on circle inversion. The power and beauty of this advanced topic is made astoundingly clear when one discovers that this problem has a tractable, unique solution based on a single application of the Pythagorean Theorem! During the MSRI demonstration, Stankova led young participants through a series of interactive challenges on the principles of Eulerian circuits and on winning strategies in some innovative checker-move games.

One thing was clear from the demonstrations: both programs have hit upon ways not only to excite young students with mathematics, but also to help young folk develop the tenacity to tackle sustained challenges via consistent—and joyful—hard work. In each circle the creative and organic mathematical process is clearly laid bare and students are placed in command of their own learning. What an accomplishment! Rick Umiker of St. Mark's School, an independent high school in Southborough, MA, comments: "Math circles demonstrate very good teaching … Are they rediscovering the power of small classes and an intimate environment?" Is it precisely the personal, intimate nature of the experience that leads to a circle's success? Is it perhaps the human experience that is being laid bare?

On this issue Shubin writes “Circles might be harmful if taught without caution and discretion, or without life and spark. And I don’t know what is worse. … As every delicate, subtle and complex organism, they [circles] require very specific and diligent care in order to thrive.” Stankova comments:

> [Math circle] sessions must be masterfully designed so as to do an array of things: invite the students, intrigue them, engage them, teach them, challenge them, and leave them with more questions to think about than when they entered the session. The format of the session is less relevant, as long as the above goals are achieved. How can a session leader keep the students’ attention on harder or more intricate concepts: that’s what distinguishes a truly gifted teacher.

One could naively say that it is not difficult to start a circle: simply gather a few young students and add a handful of exciting problems. The amount of organization and finance needed is minimal. But Paul Zeitz expresses concern that it all seems to ultimately rely on personality—and overloaded schedules. “[These] programs work because of one or two people with incredible charisma making sacrifices. There is no evidence of a program that is truly self-sustaining.” Is the only feasible math circle model a local one, run by the passion and dedication of an energetic individual or two? It seems to be the only model that currently exists in the U.S. Is this the one we should encourage and support? And if so, how do we find math circle leaders with just the right touch? How do we cultivate and support them? And can we share resources?

The Role of Competitions

Melanie Wood, a graduate student at Princeton and former Putnam fellow and International Mathematical Olympiad (IMO) silver medalist, expressed an alarming concern at the conference. She said she felt a negative bias from the research community for having succeeded so well in the competition world. “Some people reason that since problem solving isn’t ‘real math’, then students who did well in competitions must not be good at research and, in particular, decide those students don’t have the patience it takes to do research.” Melanie expressed a sentiment that her IMO colleagues also present at the conference supported, that the competition route brought her great joy and success in mathematical exploration, that she was exposed to and learned a considerable bulk of new mathematics outside the typical school curriculum, and that she developed thinking skills and maturity of mind that can only be described as an incredible asset and advantage as she now embarks on a path of original inquiry.

Joe Gallian of the University of Minnesota, Duluth, recent second vice president of the Mathematical Association of America, remarked that in his observation students who had participated in and were good at competitions are generally doing better in Research Experiences for Undergraduate programs than the typical participant. Inna Zakharevich, a winner of the USA and the Bay Area Mathematical Olympiads, and now at Harvard University, also added that students do not feel bad if they lose a competition. She stated that the general attitude is one of struggling against problems rather than competing against colleagues and that everyone appreciates and admires a good solution even if it is not one’s own.

The primary role of competitions is often perceived as a means for identifying and culling bright potential in mathematics and consequently as fundamentally elitist. Is it possible to turn this perception around and foster, articulate, and communicate instead the sentiments expressed by the young scholars? Rossi comments:

> Just look at what’s going on and observe that competition is an essential motivator for some people, and irrelevant or even detrimental to others. Is it bad to have problems drive education and good to drive education with content? Or the other way around? … Both approaches work well, especially together.

Some suggested at the conference that high-school teachers might not know what the ultimate mission of the competition experience is for their students, nor know how to prepare students for them. Can we help? Individuals, such as Richard Rusczyk, with his site `http://www.artofproblemsolving.com`, are attempting to do so. How can we support and aid such attempts? And what about those for whom the culture of competitions might be deemed “detrimental”? Are we adequately conveying multiple definitions of success in mathematics? Are we clear ourselves about the image we wish to promote?

The Typical Secondary Curriculum: Do We Have Something to Offer?

Many secondary-school teachers feel that the nature of the teenage mind is different from the mind of a young adult in college, requiring special attention and care and special approaches when it comes to education in mathematics. They are, of course, right, and the U.S. secondary educational system has, over the decades, homed in on a by and large successful, and certainly valuable, approach to mathematics education. The question is not, what is wrong with how mathematics is taught in the secondary scene, but what more can we offer? The existence of extracurricular mathematics programs is not a statement of dissatisfaction, only the recognition that there is certainly room and opportunity for discussion and connection between the pre-college and college worlds.

"We need math circles for teachers," comments Umiker, "so that they will value that kind of freedom for their students. [We need to see] the things that can be done around the edges. Not all of us are aware of what can be done."

A typical example that comes to mind is the introduction of the trigonometric functions in the ninth-grade curriculum. It comes at a time when many—but certainly not all—young teenagers are starting to learn not to memorize formulae. Yet many texts first introduce the subject as a list of three (perhaps six) ratios to be committed to memory ("SOHCAHTOA").

A math circle-type approach (as I have done with young students in a math circle) would be to introduce "circle-ometry" and define the sine and cosine (properly!) as the "height" and "overness" functions of a point rotating about the simplest circle possible in mathematics, the unit circle centered about the origin. Just to play with (and cement?) the ideas, one can then explore "square-ometry" and look at, and graph, the squine and cosquine functions, as my young students dubbed them, described by a point moving about a square with vertices $(\pm 1, \pm 1)$. What do these graphs look like? Could this also be done in typical ninth-grade classrooms? It would be interesting to find out. "The research community," adds Umiker, "should note that we need people who can challenge us to explore mathematics beyond a prescribed end."

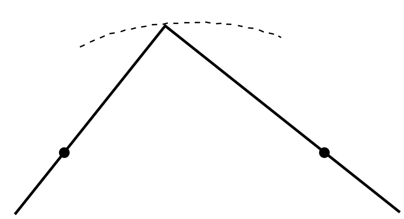
Of course, secondary educators are faced with the absolute necessity to cover a fixed bulk of content. (The pressure I personally experience in the secondary world is far greater than anything I ever felt teaching at the college level.) Math circles do not have to contend with this. Nor do secondary teachers have the luxury of working with a self-selected group of math-excited students. But these are not insurmountable issues. Tatiana Shubin is delighted to say that she is having some success in her college calculus classes moving away from center stage, and I, in my ninth-grade and AP calculus teaching, have not at all given up my math circle tendencies. Multiple approaches can successfully work together. The issue is to explore how to communicate ideas, share resources between educators of all levels, and find the forum to discuss observations and results.

Another issue to consider is the role of high-school textbooks. They are designed to be intellectually safe and are usually written so as to provide the young scholar structure, processes composed of small steps, routine, and rote practice. They also aim to provide good psychological impact—it feels good to young students to be right, and it feels good to have success quickly. Problems in textbooks are often carefully designed to offer hand-holds, pats on the back, and indicators of success. (If, for example, you find yourself working with the familiar quantity $\sin(30^0)$ chances are you must be on the right path.)

But one could note that research mathematicians and students taking competitions often look for the same indicators of success. Complicated problems are usually attacked in very small steps, and in studying them one is always looking for familiarity and connection to techniques previously practiced. If progress on a problem is leading to a certain sense of elegance, or if a formula obtained possesses symmetry of some kind, then one tends to feel good and feel confident about the path one is following. (A key difference here, of course, is that success is usually not garnered quickly. Nor are solution manuals available.)

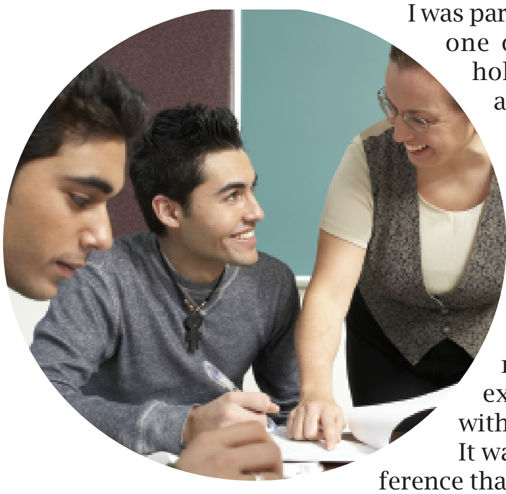
Is it possible to view the experience offered through the typical mathematics text as intimately connected to the research mathematical experience? Is this too radical a point of view? Is there a way to highlight a connection between the typical school curriculum and the creative research experience? Is this what math circles, math camps, and math competitions are ultimately trying to offer?

I was personally confronted with this issue some months ago when I came to theorem 5.15 of my class's geometry text: *The midpoint of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle.* One could, of course, present this as a known result to be proved and follow up the discussion with a variety of practice problems to be completed. I decided to turn matters around and offered instead a mystery:



Place two tacks in a wall. Insert a sheet of paper between them at some angle and mark where the corner of the paper lies. Move the paper to a different angle between the tacks and again mark the location of the corner. Repeat multiple times. What curve is produced?

In the lively discussion that ensued students discovered theorem 5.15 for themselves, proved it, and then began to wonder about other mysteries: *What if the corner of the paper is not a ninety* *degree angle: do we still obtain the arc of a circle? Is the converse true: Given a circle first, does this mean that all angles from the diameter are ninety degrees in measure?* (Note that the answer to this latter question provides a nifty means for finding the diameter of a given circle using nothing more than the corner of a piece of paper.)



I was particularly delighted that one of my students took hold of these problems and managed to prove, completely in his own way, that all points subtending the same angle from a fixed chord do indeed trace the arc of a circle—a remarkable achievement. I witnessed the math circle experience come alive within my classroom.

It was apparent at the conference that many extracurricular activities—math circle topics, competition problems—favor graph theory, combinatorics, and number theory as sources of content. (Admittedly, geometry too.) These topics are immediately accessible and offer multiple routes of exploration and discovery. Surprisingly, none of these topics appear in any depth in the typical secondary curriculum, if at all. Is it worth asking why? Are there ways to make all topics—pre-calculus? algebra II?—equally appealing and accessible, and to present them with multiple paths of discovery and exploration? Is this appropriate? How much of this is content-dependent? How much is dependent on individual teacher style? How do we connect with and support teachers who may already be asking these questions and experimenting? Is this the wrong track?

What Can We Do to Support Educators on All Levels?

Elevating mathematics through education is a noble pursuit. The work being done by those organizing and running math competitions, math camps, and math circles is often unrecognized by their supporting institutions and is done as an overload to their professional activities. Secondary-school teachers have demands placed upon them above and well beyond the requirements of simply teaching mathematics, often leading to fragmented and ridiculously lengthy work days. There is often very little freedom of mind (and freedom of practice) to pause, reflect, and experiment. Yet the determination and passion of a growing number of educators to look for and provide more is astounding. What can be done to offer support?

For me, discovering that I am not alone in this pursuit was a great comfort. Establishing connections between like-minded educators is proving to be immensely fruitful and rewarding. Discussions about the issues raised carry on through email and local discussions, and ideas and approaches are actively being explored. I am delighted that this was the *first* of a series of conferences on the topic of Math Circles and Olympiads that MSRI intends to offer. The number of attendees surely will grow.

Fundamental questions remained unanswered. Some participants wonder, for example, whether the math circle model is destined to remain localized and extracurricular. Others are trying to incorporate math circle ideals directly into the classroom experience. Work is under way to create a general website, supported by MSRI, offering advice, plans, and resource materials as a means to reach out to those who may be interested in exploring these ideas. We should consider how to help teachers pursue this work. Would the formation of a special interest group on math circles and competitions through the mathematics professional societies be of help?

Mary Fay-Zenk, mathematics resource teacher for the Cupertino Union School District, CA, having experienced great pedagogical success with the use of math competitions as motivators, suggested the idea of starting a math circle for middle- and high-school teachers. A number of people are picking up on this idea. Matthias Beck of San Francisco State University and Paul Zeitz of the University of San Francisco, for instance, with the support of MSRI and funded by the McKesson Foundation, are starting a new math circle for young students in the Bay Area, accompanied with math-circle type courses for their teachers. St. Mark's Institute of Mathematics in Southborough, MA, in collaboration with the Northeastern University School of Education, began a similar program for teachers. Not only will programs like these disseminate the ideas and principles of math circle teaching, they will also help establish a network of support and communication.

MSRI is organizing a number of special workshops to explore and address directly some of the questions raised at its December 2004 conference. For example, the "Mathematical Knowledge for Teaching" workshop, May 25–28, 2005, brought together K–12 educators, educational researchers, mathematicians, and policy makers to examine what is known about the knowledge needed for teaching mathematics.

Is this enough to get the big ideas "out there"? Perhaps the only route for success along these lines is to consistently offer forums for discussion and to rely on local dissemination of ideas. (Such a route certainly works for the Boston Math Circle, for instance. Relying solely on word-of-mouth, the program is consistently over 120 students strong.)

Some programs are reluctant to write down any form of "curriculum" to share, not because of a reluctance to disseminate ideas—far from it—but rather because the very nature of creative exploration is organic and nonlinear and cannot be prescribed. On the other hand, one can argue that it is certainly better to have something written to share, even if it ends up not being used as intended. (And thankfully the Kaplans have decided on this too. Their book, *Out of the Labyrinth: Mathematics Set Free* will be released by Oxford University Press later this year.) I have to say that Dmitri Fomin, Sergey Genkin, and Ilia Itenberg's book *Mathematical Circles (Russian Experience)* (AMS, Providence, 1996) significantly guided my own ideas about how to conduct a math circle. (For the more structured experience, one may consult [7] for a collation of over seven years' worth of Berkeley Math Circle lecture notes and [3] for an impressive collection of guest lectures given at the Bay Area Mathematical Adventures series, a program intimately connected with the San Francisco Bay Area math circles.)

Maybe the key is to articulate and clarify the notion that mathematics is ultimately a creative human endeavor, and maybe we should strive to offer means for it to be experienced as such by teacher and student alike. Can we communicate (teach?) educators of all levels not to fear pursuing, or at least exploring, the creative process—to let go of the perception of needing to be the master of the subject at all times? Can we encourage folks to trust the mathematical experience even if one cannot identify where a class is heading with it at a particular moment (day? week?)? Can we encourage educators to be comfortable and confident with the process even if time is running out, there is a common exam next week, and the Dean of Faculty wants evidence of demonstrable success? Should we?

References

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[6] The Berkeley Math Circle: `http://mathcircle.berkeley.edu`.

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[8] The San Jose Math Circle: `http://www.mathcircles.org`.