Mushroom Billiards

Mason A. Porter and Steven Lansel

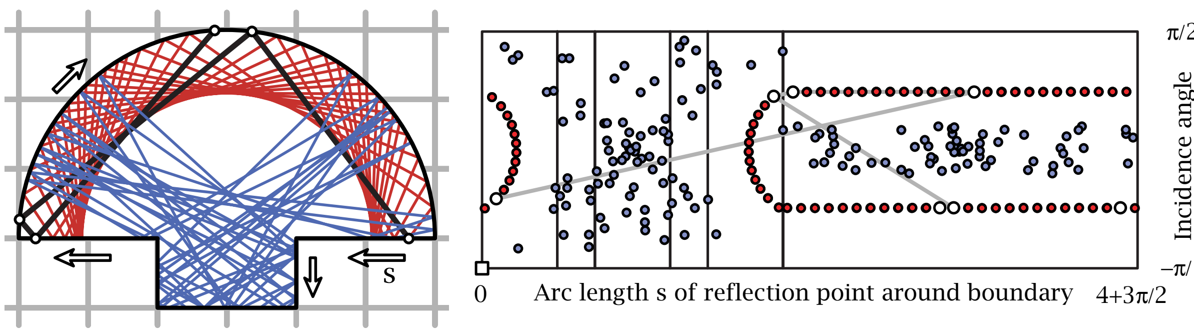

Figure 1. (Left) Axially symmetric mushroom billiard with a semicircular cap. The red trajectory lies in the integrable region of phase space (the ambient space of positions and velocities), and the blue one lies in the chaotic region. (Right) Billiard map showing discretizations of these trajectories. The vertical lines delineate singular points (corners) between different arcs of the mushroom.

Two 20th century discoveries transformed scientists' understanding of nonlinear phenomena [16]. One was Kolmogorov-Arnold-Moser (KAM) theory, which demonstrated the stability of regular dynamics for small perturbations of Hamiltonian systems [4, 5, 11, 15]. The other was the theory of stochasticity of dynamical systems (loosely called "chaos theory"), which demonstrated the stability of strongly irregular dynamics under small perturbations [3, 18, 20]. To gain a better understanding of complicated nonlinear dynamics, there have been extensive studies of model systems such as billiards.

The mathematical idealization of a (classical) billiard, a ubiquitous Hamiltonian system [9, 21], consists of a confined point particle colliding elastically against the boundaries of a container of some shape [17, 19]. Like their real-life namesakes, these sorts of billiards have long held the interest of both mathematicians and physicists. They have led to numerous advancements in ergodic theory and dynamical systems and can even be constructed in experimental settings such as microwave cavities [12] and atom optics [10].

Geometrically, an orbit of the *billiard flow* of a confined particle is described by a union of line segments, with the link after a boundary collision determined according to the rule that the angle of incidence equals the angle of reflection. Keeping track only of the location and incidence angles of the collisions gives a discrete-time *billiard map*.

The best-known examples of *chaotic* billiards, whose flow is hyperbolic, ergodic, mixing, and Bernoulli [17, 19], are the "dispersing" Sinai billiard (a square table with a circular barrier at its center) and the Bunimovich stadium (shaped like a rectangle with two "focusing" circular caps). Neighboring parallel orbits diverge when they collide with dispersing components of a billiard's boundary. In chaotic focusing billiards, neighboring parallel orbits converge at first, but divergence prevails over convergence on average. Divergence and convergence are balanced in *integrable* billiards such as circles and ellipses, the position spaces ("configuration spaces") of which are continuously foliated by one or more families of "caustics". A curve is called a caustic if whenever any link of some trajectory is tangent to it, then all other links of the same trajectory are also tangent to it.

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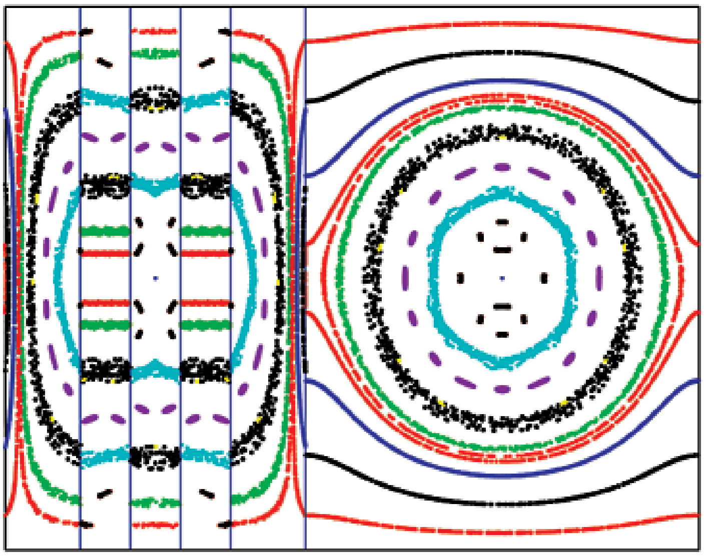


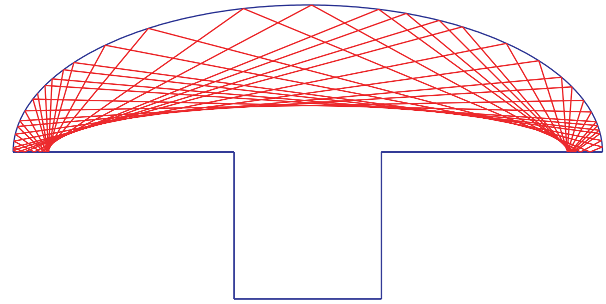


Figure 2. (Left) Billiard map for elliptical mushroom with right:left cap ratio of 1:1. Islands of integrability arise from the presence of elliptical caustics (Right). (The curves at the top and bottom of the rightmost region of phase space, which can also be seen in two other regions, are due to the elliptical caustics.)

Fully chaotic and integrable systems are atypical, as most Hamiltonian systems (not just billiards) are actually *mixed*, with islands of integrability situated in a "chaotic sea" formed by one or more ergodic components that occupy a subset of positive measure (volume) of the system's ambient space of positions and velocities (called its "phase space"). Mixed Hamiltonian systems are notorious for being difficult to analyze rigorously, as the approaches developed for completely integrable and chaotic systems both fail at the boundaries between chaotic and regular regions. Computational investigations of mixed systems are similarly difficult, as small islands are not easy to find numerically.

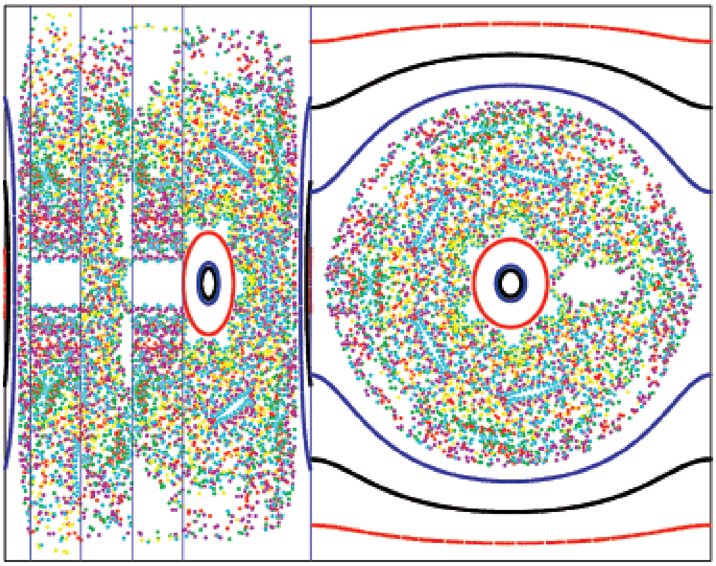
In 2001, Leonid Bunimovich introduced a class of billiard containers shaped like idealized mushrooms [6] (reminiscent of the kind many of us saw in certain 1980s video games), providing a class of examples with mixed regular-chaotic dynamics whose relatively simple geometry makes precise analysis feasible. This discovery has thus made it possible to address some delicate questions about the dynamics of systems with coexisting islands and chaotic regions [1, 6, 7].

The simplest mushroom billiards (see Figure 1), consisting of a semicircular cap with a stem of some shape attached to the cap's base, provide a continuous transition between integrable (semi)circular billiards and ergodic (semi)stadium billiards as the stem width is increased from zero to the diameter of the circle. There are a large variety of admissible stem shapes, but we'll stick to rectangular ones here.

Bunimovich proved that for circular mushrooms, trajectories that remain in the cap are integrable, whereas those that enter the stem are chaotic (except for a set of measure zero). This gives a precise, complete characterization of mushroom billiard trajectories. One can take advantage of this knowledge in several ways. Changing the dimensions of a mushroom billiard controllably alters the relative volume fractions in phase space of initial conditions leading to integrable and chaotic trajectories. Using these mushrooms, Bunimovich has also shown that the stationary distribution of noninteracting particles in a container can be nonuniform [7]. The recent study of two, three, and four finite-radius disks in a circular container has shown that this can also happen with interacting particles [14].

Although they do not have the KAM island hierarchies of generic mixed Hamiltonian systems, mushroom billiards possess "sticky" chaotic trajectories, characterized by long tails in recurrence-time statistics—they have a power-law distribution rather than an exponential one—that experience long periods of almost regular motion [1, 2]. The simple geometry of mushrooms allows the analytical study of such statistics, which are known to strongly influence global properties of Hamiltonian systems, including correlation decay and phase-space transport. The long tails arise from a zero-measure family of angle-preserving "marginally unstable periodic orbits", with zero Lyapunov exponent and real eigenvalues of modulus one, that lie inside the chaotic region but never visit the mushroom stem.

Mushrooms with elliptical caps have one chaotic component and either zero, one, or two islands when the stem is sufficiently long. (Extra KAM islands arise when the stem is too short, as one can see especially clearly in Figure 2.) This statement was proven in [6] and is illustrated by Figures 2–4, which show what happens to the elliptical mushroom's billiard map as the rectangular stem is shifted from left to right. One island of integrability, associated with trajectories tangent to the cap's elliptical caustics, exists if and only if the stem does not intersect the edges of the cap (see Figures 2 and 3) and the other, due to trajectories tangent to its hyperbolic caustics, exists if and only if the stem does not contain the center of the cap's base (see



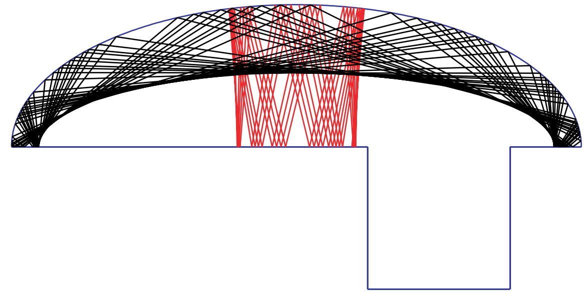


Figure 3. Right:left cap ratio of 1:5. Islands arise from both elliptical and hyperbolic caustics (Right). (The central curves in the right two regions of phase space are due to the hyperbolic caustics.)

Figures 3 and 4) [6]. By contrast, circular mushrooms contain no islands if the stem intersects the edge of the cap and do not require the extra condition concerning the stem length (no KAM islands arise from the stem being short). The difference when the stem intersects the edge of the cap arises because elliptical billiards have both elliptical and hyperbolic families of caustics and circular billiards have only the elliptical one. As the stem is shifted to the right, the volume in phase space occupied by the elliptical island shrinks, ultimately disappearing when the stem reaches the edge of the cap. The orbits that were tangent to elliptical caustics now enter the stem, making the top and bottom of phase space chaotic. There is still an island in the center corresponding to orbits tangent to hyperbolic caustics that do not enter the stem.

Generalizations of mushroom billiards have also been studied. One can construct billiards with arbitrarily many integrable and chaotic components, each encompassing an arbitrary fraction of the phase space volume, using so-called "honey mushrooms", which have multiple stems and caps [6].Three-dimensional generalizations of mushroom billiards have also been examined [8]. Finally, the eventual study of quantum mushroom billiards, whose dynamics satisfy the Schrödinger equation with homogeneous Dirichlet boundary conditions, should shed considerable light on the quantization of systems with mixed dynamics.

Acknowledgements

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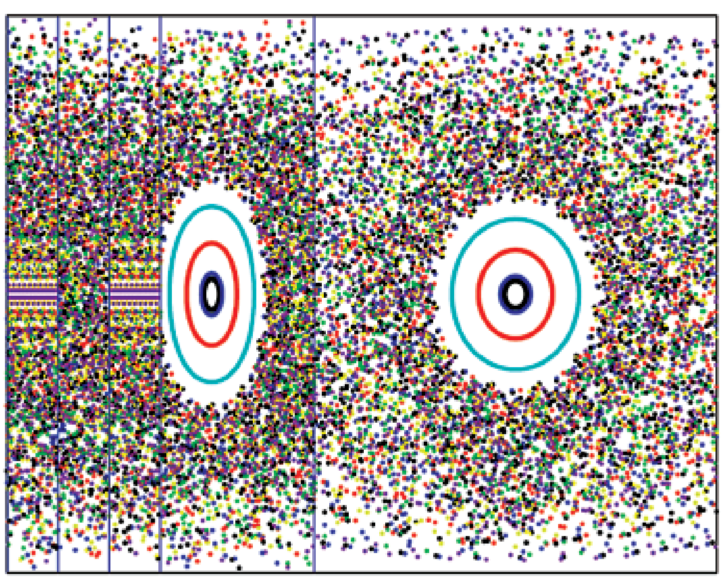

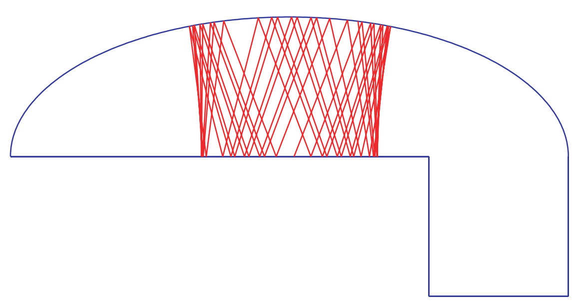

Figure 4. Right:left cap ratio of 0. That is, the right end of the stem touches the edge of the elliptical cap. Islands arise from hyperbolic caustics (Right).

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About the Cover

Phase Portrait of a Mushroom

The cover shows a close-up of part of the phase space diagram of one of the "mushrooms" described by Mason Porter and Steven Lansel in this issue. The full diagram is shown below, and what the cover shows of this is at the bottom.

—*Bill Casselman, Graphics Editor*
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