Gödel, Inconsistency, Provability, and Truth: An Exchange of Letters

On the eve of Gödel's centennial, it is distressing that his incompleteness theorems continue to be so misunderstood. The article "Whither Mathematics?", which appeared in the December 2005 *Notices*, is an egregious example. On page 1350 it is stated that Gödel "established that the consistency of arithmetic was not provable", a claim reiterated again four pages later. But Gödel did no such thing. Rather, he proved that if formalized arithmetic is consistent, a particular numerical encoding of that fact is expressible, but not provable, within that theory itself. The theorem does not rule out persuasive proofs of the consistency of arithmetic that employ means not formalizable within arithmetic. The first such proof was given by Gerhard Gentzen in 1936, and Gödel himself outlined another in his last published paper, which appeared in 1958.

The (second) incompleteness theorem showed that Hilbert's proof theory, which aimed to demonstrate the consistency of mathematics by a bootstrapping process, was incapable of realization. But the theorem is irrelevant to those (such as the article's author) who think that Peano arithmetic might be inconsistent; for if it is, every statement expressible within it, including the encoded assertion of its consistency, will be provable therein. So Hilbert's aim only made sense (even before Gödel's theorem) for those who believed that formalized arithmetic was consistent.

Nor do I understand what the author means when he says that "Godel's theorems…do not establish that there is a fundamental distinction between truth and provability in mathematics without the insertion of extra philosophical assumptions." There is a precise characterization (due to Tarski, shortly after the appearance of the incompleteness theorems) of what it means for a statement in a formal language to be true within a given structure for that language. That characterization is independent of any syntactic considerations, and in the case of formalized arithmetic it is not expressible within the theory itself (unlike the notion of a proof in that theory). It is true that Gödel himself did not demonstrate that inexpressibility (though there is ample evidence that he was aware of it before Tarski's work). Rather, acutely conscious of how controversial the idea of an objective notion of mathematical truth then was, he invoked the notion of truth only in the informal introduction to his incompleteness paper, where he "sketch[ed] the main idea of the proof …without any claim to complete precision." In the main part of the paper he eschewed semantic considerations altogether, employing only methods that were acceptable to formalists (whose program he had initially set out to advance).

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Davies Response

Professor's Dawson's criticisms of the small part of my article "Whither Mathematics" referring to Gödel's work shows how easy it is to annoy and possibly mislead people by over-abbreviation. He correctly states that I should have said that "the consistency of arithmetic is not provable *from within arithmetic*" on page 1350 of my article. (I failed to repeat this essential caveat, contained in the previous sentence of my article.) However, if one proves the consistency of arithmetic by invoking some other, richer, formal system, one achieves nothing unless one considers that the consistency of that new system is less capable of being doubted. Gentzen's proof of consistency, for example, uses transfinite induction, and was not regarded as persuasive by Tarski. Angus Macintyre ("Mathematical significance of proof theory", *Phil. Trans. Royal. Soc.* A **363** (2005), 2419–2435, p. 2426) argues that in Gentzen's work consistency is not really the main issue at all.

I am not sure why Professor Dawson states "So Hilbert's aim only made sense for those who believed that formalized arithmetic was consistent." Surely the goal of his research programme should have been even more interesting to someone who was prepared to admit some doubt on the matter, however small, than to the great majority who thought that it was merely a formal exercise? I know of nobody who actively believes that Peano's axioms are inconsistent, but Jack Schwartz goes even further than I do in arguing that consistency is by no means well established; his article "Do the integers exist? The unknowability of arithmetic consistency", *Comm. Pure and Appl. Math.* **58** (2005), 1280–1286, is highly relevant to this matter.

The relationship between truth and provability is interesting, but I had not wanted to delve into such a controversial issue in what was principally a forward-looking article. One can certainly specify a precise technical notion of truth within a formal context, as Tarski did, but I hope that most readers did not confuse my use of the word truth with that of Tarski, who was careful to avoid making claims about the philosophical status of the term as he used it; see the polemical section of "The semantic conception of truth and the foundations of semantics", *Phil. and Phenom. Res.* **4** (1944), 13–47. While Gödel may have been acutely aware of how controversial the notion of unconditional, or Platonic, truth was, he seemed to believe that there was such an independent notion of truth in 1947. I quote

> But, despite their remoteness from sense experience, we do have something like a perception of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e. in mathematical intuition, than in sense perception. (Gödel, K., "What is Cantor's Continuum Problem?", 1947, pp. 258–273 in Philosophy of Mathematics: Selected Readings, Second Edition, (Benacerraf, Paul, and Putnam, Hilary, eds.), Cambridge University Press, New York, NY, 1983).

His use of the word "true" here cannot be that of Tarski. Both Schwartz (op. cit.) and the author (*Science in the Looking Glass*, Oxford Univ. Press, 2003, Chap. 1, 2) have argued that Gödel's argument is not tenable, for similar reasons. Paul Cohen rejected Gödel's Platonist view of set theory in "Comments on the foundations of set theory", *Axiomatic Set Theory*, Proceedings of Symposia in Pure Mathematics, (D. Scott, ed.), Vol. 13, Part 1, Amer. Math. Soc., Providence, RI, pp. 9–15, concluding that it "is our fate, to live with doubts, to pursue a subject whose absoluteness we are not certain of, in short to realize that the only 'true' science is itself of the same mortal, perhaps empirical, nature as all other human undertakings" (p. 15). In "Skolem and pessimism about proof in mathematics", *Phil. Trans. Royal. Soc.* A **363** (2005), 2407–2418, Cohen reiterates his doubts, particularly about the relationship between axiom systems involving large cardinals and reality (p. 2416). I used the phrase "philosophical assumptions" to refer to this issue, although it seems to provoke some people. I am less persuaded that Peano arithmetic is necessarily and obviously consistent than most mathematicians, but I do not lose any sleep over the possibility that my life's work will one day be suddenly rendered invalid. The ingenuity of mathematicians is enormous, and any internal contradictions would probably be overcome fairly quickly with minimal effects on the overall body of mathematics.

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