WHAT IS...

Percolation?

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Percolation is a simple probabilistic model which exhibits a phase transition (as we explain below). The simplest version takes place on $\mathbb{Z}^2$, which we view as a graph with edges between neighboring vertices. All edges of $\mathbb{Z}^2$ are, independently of each other, chosen to be *open* with probability p and *closed* with probability $1-p$. A basic question in this model is "What is the probability that there exists an open path, i.e., a path all of whose edges are open, from the origin to the exterior of the square $S_n := [-n, n]^2$?" This question was raised by Broadbent in 1954 at a symposium on Monte Carlo methods. It was then taken up by Broadbent and Hammersley, who regarded percolation as a model for a random medium. They interpreted the edges of $\mathbb{Z}^2$ as channels through which fluid or gas could flow if the channel was wide enough (an open edge) and not if the channel was too narrow (a closed edge). It was assumed that the fluid would move wherever it could go, so that there is no randomness in the behavior of the fluid, but all randomness in this model is associated with the medium.

We shall use $\mathbf{0}$ to denote the origin. A limit as $n \to \infty$ of the question raised above is "What is the probability that there exists an open path from $\mathbf{0}$ to infinity?" This probability is called the *percolation probability* and denoted by $\theta(p)$. Clearly $\theta(0) = 0$ and $\theta(1) = 1$, since there are no open edges at all when $p = 0$ and all edges are open when $p = 1$. It is also intuitively clear that the function $p \mapsto \theta(p)$ is nondecreasing. Thus the graph of θ as a function of p should have the form indicated in Figure 1, and one can define the *critical probability* by $p_c = \sup\{p : \theta(p) = 0\}$.

Why is this model interesting? In order to answer this we define the *(open) cluster* $C(v)$ of the vertex $v \in \mathbb{Z}^2$ as the collection of points connected to v by an open path. The clusters $C(v)$ are the maximal connected components of the collection of open edges of $\mathbb{Z}^2$, and $\theta(p)$ is the probability that $C(\mathbf{0})$ is infinite. If $p < p_c$, then $\theta(p) = 0$ by definition, so that $C(\mathbf{0})$ is finite with probability 1. It is not hard to see that in this case all open clusters are finite. If $p > p_c$, then $\theta(p) > 0$ and there is a strictly positive probability that $C(\mathbf{0})$ is infinite. An application of Kolmogorov's zero-one law shows that there is then with probability 1 some infinite cluster. In fact, it turns out that there is a unique infinite cluster. Thus, the global behavior of the system is quite different for $p < p_c$ and for $p > p_c$. Such a sharp transition in global behavior of a system at some parameter value is called a *phase transition* or a *critical phenomenon* by statistical physicists, and the parameter value at which the transition takes place is called a critical value. There is an extensive physics literature on such phenomena. Broadbent and Hammersley proved that $0 < p_c < 1$ for percolation on $\mathbb{Z}^2$, so that there is indeed a nontrivial phase transition. Much of the interest in percolation comes from the hope that one will be better able to analyze the behavior of various functions near the critical point for the simple model of percolation, with all its built-in independence properties, than for other, more complicated models for disordered media. Indeed, percolation is the simplest one in the family of the so-called random cluster or Fortuin-Kasteleyn models, which also includes the celebrated Ising model for magnetism. The studies of percolation and random cluster models have influenced each other.

Percolation can obviously be generalized to percolation on any graph $\mathcal{G}$, even to (partially) directed graphs. One can also consider the model in which the vertices are independently open or closed, but all edges are assumed open. This version is called

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site percolation, in contrast to the version we considered so far, and which is called *bond percolation*. Initially research concentrated on finding the precise value of p_c for various graphs. This has not been very successful; one knows p_c only for a few planar lattices (e.g., $p_c = 1/2$ for bond percolation on $\mathbb{Z}^2$ and for site percolation on the triangular lattice). The value of p_c depends strongly on geometric properties of $\mathcal{G}$. Attention has therefore shifted to questions about the distribution of the number of vertices in $C(\mathbf{0})$ and geometric properties of the open clusters when p is close to p_c. It is believed that a number of these properties are *universal*, that is, they depend only on the dimension of $\mathcal{G}$, and not on details of its structure.

In particular, one wants to study the behavior of various functions as p approaches p_c, or as some other parameter tends to infinity, while p is kept at p_c. It is believed that many functions obey so-called *power laws*. For instance, it is believed that the expected number of vertices in $C(\mathbf{0})$, denoted by $\chi(p)$, behaves like $(p_c - p)^{-\gamma}$ as $p \uparrow p_c$, in the sense that $-\log \chi(p)/\log(p_c - p) \to \gamma$ for a suitable constant γ. Similarly one believes that $\theta(p)$ behaves like $(p - p_c)^{\beta}$ for some β as $p \downarrow p_c$, or that the probability that there is an open path from $\mathbf{0}$ to the exterior of S_n for $p = p_c$ behaves like $n^{-1/\rho}$ for some ρ. Even though such power laws have been proven only for site percolation on the triangular lattice or on high-dimensional lattices, it is believed that the exponents β, γ, ρ, etc. (usually called *critical exponents*), exist, and in accordance with the universality hypothesis mentioned above depend only on the dimension of $\mathcal{G}$. For instance, bond and site percolation on $\mathbb{Z}^2$ or on the triangular lattice should all have the same exponents. Physicists invented the renormalization group to explain and/or prove such power laws and universality, but this has not been made mathematically rigorous for percolation.

$\mathbb{Z}^d$ for large d behaves in many respects like a regular tree, and for percolation on a regular tree one can easily prove power laws and compute the relevant critical exponents. For bond percolation on $\mathbb{Z}^d$ with $d \ge 19$ Hara and Slade succeeded in proving power laws and in showing that the exponents agree with those for a regular tree. They have even shown that their theory applies down to $d > 6$ when one adds edges to $\mathbb{Z}^d$ between any two sites within distance L_0 of each other for some $L_0 = L_0(d)$.

Due to this theory we have a reasonable understanding of high-dimensional percolation. In the last few years Lawler, Schramm, Smirnov, and Werner have proven power laws for site percolation on the triangular lattice and confirmed most of the values for the critical exponents conjectured by physicists. Their proof rests on Schramm's invention of the Stochastic Loewner Evolutions or Schramm Loewner Evolutions (SLE) and on Smirnov's beautiful proof of the existence and conformal invariance properties of certain crossing probabilities. Roughly speaking this says the following: Let $\mathcal{D}$ be a "nice" domain in $\mathbb{R}^2$ and let A and B be two arcs in the boundary. For $\lambda > 0$, let $P_\lambda(\mathcal{D}, A, B)$ be the probability for $p = p_c$ that there exists an open path of site percolation on the triangular lattice in $\lambda\mathcal{D}$ from λA to λB. In fact it is neater to take $P_\lambda(\mathcal{D}, A, B)$ as the probability at p_c of an open connection in $\mathcal{D}$ from A to B on $(1/\lambda)$ times the triangular lattice. Conformal invariance says that $Q(\mathcal{D}, A, B) := \lim_{\lambda\to\infty} P_\lambda(\mathcal{D}, A, B)$ exists and that $Q(\mathcal{D}, A, B) = Q\big(\Phi(\mathcal{D}), \Phi(A), \Phi(B)\big)$ for every conformal map Φ from $\mathcal{D}$ onto $\Phi(\mathcal{D})$. Further crucial ingredients are characterizations by Lawler and Werner of some SLE process on a domain by means of properties of its evolution before it hits the boundary. Conformal invariance had earlier been conjectured by physicists, and Cardy had given a formula for $Q(\mathcal{D}, A, B)$. Smirnov's work gives a rigorous proof of Cardy's formula for percolation on the triangular lattice. Further work (see Camia and Newman (2005)) also has led to a description of the limit (in a suitable sense) as $\lambda \to \infty$ of the full pattern of the random configuration of open paths at criticality, i.e., for $p = p_c$. Since their discovery, SLE processes have led to exciting new probability theory in their own right, for instance, to power laws for the intersection probabilities of several Brownian motions (see Lawler (2005)).

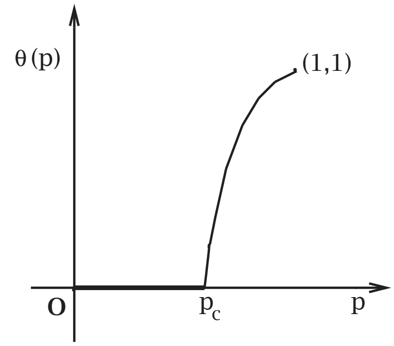

Figure 1. Graph of θ. Many aspects of this graph are still conjectural.

So far conformal invariance results have been achieved only for site percolation on the triangular lattice. It is perhaps the principal open problem of the subject to prove conformal invariance for percolation on other two-dimensional lattices. Another related major problem is to establish power laws and universality for percolation on d-dimensional lattices with $2 \le d \le 6$. Finally, an unsolved problem of fifteen years' standing is whether there is an infinite open cluster for critical percolation on $\mathbb{Z}^d, d \ge 3$.

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Further Reading

[1] Federico Camia and Charles M. Newman, The full scaling limit of two-dimensional critical percolation, arXiv:math.PR/0504036.

[2] Geoffrey Grimmett, *Percolation*, second edition, Springer, 1999.

[3] Gregory F. Lawler, *Conformally Invariant Processes in the Plane*, Amer. Math. Soc., 2005.