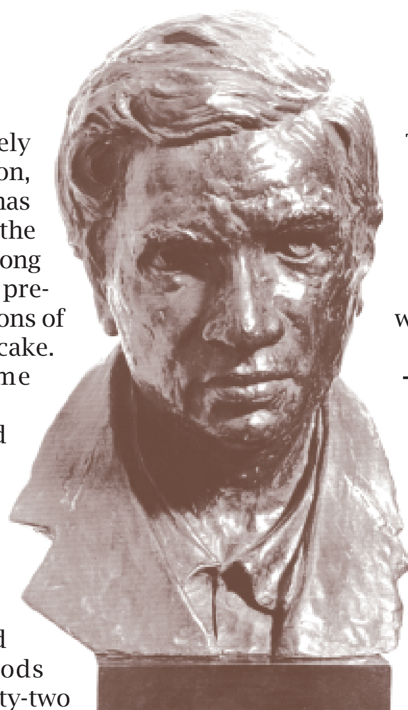
Honoring a Gift from Kumbakonam

Ken Ono



Bust of Srinivasa Ramanujan by artist Paul Granlund.

Today was an absolutely glorious day in Madison, Wisconsin. It is Christmas 2005, and everyone in the house is asleep after a long day of enjoying family, opening presents, and eating enormous portions of mashed potatoes and yule log cake. Yet powerful images keep me awake.

Thirty-six hours ago I returned from a six-day whirlwind journey to a far-off place. I spent forty hours on airplanes, and I endured fourteen hours in cars dodging bicycles, rickshaws, cows, goats, and masses of people on roads severely damaged by recent flooding. These floods would be blamed[1] for at least forty-two deaths. Despite these hardships and bad luck, this adventure exceeded my lofty expectations.

I ostensibly travelled to Kumbakonam with the purpose of giving a lecture on mock theta functions and Maass forms at the International Conference on Number Theory and Mathematical Physics at SASTRA University. I could have offered other worthy pretexts: I wanted to see my student Karl Mahlburg give his first plenary lecture. I wanted to applaud my friends Manjul Bhargava and Kannan Soundararajan (he goes by Sound) as they won a prestigious prize. However, my primary reason was personal, not professional.

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[1] *This was reported in* The Hindu *on December 20, 2005.*

This adventure was a pilgrimage to pay homage to Srinivasa Ramanujan, the Indian legend whose congruences, formulas, and identities have inspired much of my own work. This fulfilled a personal journey, one with an unlikely beginning in 1984.

The Story of Ramanujan

Ramanujan was born on December 22, 1887, in Erode, a small town about 250 miles southwest of Chennai (formerly known as Madras). He was a *Brahmin*, a member of India's priestly caste, and as a consequence he lived his life as a strict vegetarian.

When Ramanujan was one year old, he moved to Kumbakonam, a small town about 170 miles south of Chennai, where his father Srinivasa was a cloth merchant's clerk. Kumbakonam, which is situated on the banks of the sacred Kaveri River, was (and remains today) a cosmopolitan center of the rural Indian district of Tanjore in the state of Tamil Nadu. Thanks to the area's rich soil and tropical climate, rice and sugar cane crops thrive. In Ramanujan's day, Kumbakonam had a population of fifty thousand.

Kumbakonam is one of India's sacred Hindu towns. It boasts seventeen Hindu temples (eleven honoring the Hindu god Lord Siva, and six honoring the god Lord Vishnu). The town is perhaps most well-known for its Mahamaham Festival, which is held every twelve lunar years when the Sun enters the constellation of Aquarius and Jupiter enters Leo. Nearly one million Hindu pilgrims descend on Kumbakonam for the festival. In a ritual

meant to absolve sins, pilgrims bathe in the Mahamaham tank, which symbolizes the waters of India's holy rivers.

As a young boy, Ramanujan was a stellar student. He entered Town High School in 1898, and he would go on to win many awards there. He was a strong student in all subjects, and he stood out as the school's best math student. His life took a dramatic turn when a friend loaned him the Government College library's copy of G. S. Carr's *Synopsis of Elementary Results in Pure Mathematics*. G. H. Hardy, the celebrated Cambridge professor, later described (see page 3 of [17]) the book as

> …the "synopsis" it professes to be. It contains enunciations of 6,165 theorems, systematically and quite scientifically arranged, with proofs which are often little more than cross-references…

Ramanujan became addicted to mathematics research, and he recorded his findings in notebooks, imitating Carr's format. He typically offered no proofs of any kind. Based on his education, he presumably did not understand the obligation mathematicians have for justifying their claims with proofs.

Thanks to his exemplary performance at Town High School, Ramanujan won a scholarship to Government College. However, by the time he enrolled there in 1904, his addiction to mathematics made it impossible for him to focus on schoolwork. He unceremoniously flunked out. He would later get a second chance, a scholarship to attend Pachaiyappa's College in Madras. However, mathematics again kept him from his schoolwork, and he flunked out a second time.

By 1907, the gifted Ramanujan was an academic failure. There was no room for him in India's system of higher education. Despite his failures, his friends and parents supported him. They must have recognized his genius, for they allowed him to work on mathematics unabated. Vivid accounts portray Ramanujan hunched over his slate on the porch of his house and in the halls of Sarangapani Temple, working feverishly.

> ….Ramanujan would sit working on the pial (porch) of his house on Sarangapani Street, legs pulled into his body, a large slate spread across his lap, madly scribbling, …When he figured something out, he sometimes seemed to talk to himself, smile, and shake his head with pleasure.
>
> R. Kanigel (see page 67 of [20])

It is said (for example, [3, 20]) that Ramanujan believed that his findings were divine, told to him in dreams by Namagiri, the goddess of Namakkal.

In July 1909, Ramanujan married nine-year-old S. Janaki Ammal; it was an arranged marriage. After a short stay with Ramanujan and his family, Janaki returned to her home to learn domestic skills and pass time until she reached puberty. Ramanujan moved to Madras in 1911 and Janaki joined him in 1912 to begin their married life. To support them, Ramanujan took a post as a clerk in the accounting department of the Madras Port Trust.

Ramanujan continued his research in near isolation. His job at the Port Trust provided a salary and left time for mathematics. Despite these circumstances, his frustration mounted. Although some Indian patrons acknowledged his genius, he was unable to find suitable mentors. Indian mathematicians did not understand his work.

After years of such frustration, Ramanujan boldly wrote distinguished English mathematicians. He first wrote H. F. Baker, and then E. W. Hobson, both times without success. His letters consisted mostly of bare statements of formal identities, recorded without any indication of proof. Due to his lack of formal training, he claimed some known results as his own, and he offered others, such as his work on prime numbers, which were plainly false. In this regard, Hardy would later write (see page xxiv of [16]):

> Ramanujan's theory of primes was vitiated by his ignorance of the theory of a complex variable. It was (so to say) what the theory might be if the Zeta-function had no complex zeroes. …Ramanujan's Indian work on primes, and on all the allied problems of the theory, was definitely wrong.

Ramanujan's work on Bernoulli numbers, which he presumably included in his letters, also includes an incredible mistake involving explicit numbers. The Bernoulli numbers [23] are the rational numbers $B_2 = 1/6, B_4 = 1/30, \ldots$ defined[2] by

$$x \cot x = 1 - \frac{B_2}{2!}(2x)^2 - \frac{B_4}{4!}(2x)^4 - \frac{B_6}{6!}(2x)^6 - \cdots.$$

Ramanujan falsely conjectured (see equation (14) of [23]) that if n is a positive even number, then the numerator of B_n/n, when written in lowest terms, is prime.[3] This conjecture is false, as is plainly seen by

$$\frac{B_{20}}{20} = \frac{174611}{6600} = \frac{283 \times 617}{2^3 \times 3 \times 5^2 \times 11}.$$

[2] *This is a slight departure from the modern definition of the Bernoulli numbers b_{2n}. These numbers are related by the relation $B_{2n} = (-1)^{n+1} b_{2n}$.*

[3] *Ramanujan obviously considered 1 to be a prime for this conjecture.*

In fact, among the even numbers n less than 2000, Ramanujan's conjecture holds only for the twenty numbers

$$2,4,6,8,10,12,14,16,18,26,34,36,$$
$$38,42,74,114,118,396,674,1870.$$

In view of these facts, it is not surprising that Baker and Hobson dismissed him as a crank.

Then on January 16, 1913, Ramanujan wrote G. H. Hardy, a thirty-five year old analyst and number theorist at Cambridge University. With his letter he included nine pages of mathematical scrawl. C. P. Snow elegantly recounted (see pages 30-33 of [18]) Hardy's reaction to the letter:

> One morning in 1913, he (Hardy) found, among the letters on his breakfast table, a large untidy envelope decorated with Indian stamps. When he opened it…he found line after line of symbols. He glanced at them without enthusiasm. He was by this time…a world famous mathematician, and…he was accustomed to receiving manuscripts from strangers. …The script appeared to consist of theorems, most of them wild or fantastic… There were no proofs of any kind… A fraud or genius? …is a fraud of genius more probable than an unknown mathematician of genius? …He decided that Ramanujan was, in terms of…genius, in the class of Gauss and Euler…

Hardy could have easily dismissed Ramanujan like Baker and Hobson before him. However, to his credit he (together with Littlewood) carefully studied Ramanujan's scrawl and discovered hints of genius. In response to Ramanujan's letter, Hardy invited Ramanujan to Cambridge for proper training. Although Hindu beliefs forbade such travel at the time, we are told that Komalatammal, Ramanujan's mother, had a vision from the Hindu Goddess Namagiri giving Ramanujan permission to accept Hardy's invitation. Ramanujan accepted, and he left his life in south India for Cambridge, home of some of the world's most distinguished scientists and mathematicians. He arrived on April 14, 1914.

Over the course of the next five years, Ramanujan would publish extensively on a wide variety of topics: the distribution of prime numbers, hypergeometric series, elliptic functions, modular forms, probabilistic number theory, the theory of partitions and q-series, among others. He would write over thirty papers, including seven with Hardy. After years of frustration working alone in India, Ramanujan was finally recognized for the content of his mathematics. He was named a Fellow of Trinity College, and he was elected a Fellow of the Royal Society (F.R.S.), an honor shared by Sir Isaac Newton. News of his election spread quickly, and in India he was hailed as a national hero.

Ramanujan grew ill towards the end of his stay in England. One of the main reasons for his declining health was malnutrition. He was a vegetarian living in World War I England, a time when almost no one else was a vegetarian. Ramanujan also struggled with the severe change in climate; he was not accustomed to English weather. He did not have (or did not wear) appropriate clothes to protect himself from the elements. These conditions took their toll, and Ramanujan became gravely ill. He was diagnosed with tuberculosis. More recently, hepatic amoebiasis [4, 29], a parasitic infection of the liver, has been suggested as the true cause of his illness.

Hardy would visit the bedridden Ramanujan at a nursing home in Putney, a village a few miles from London on the south bank of the Thames.

> It was on one of those visits that there happened the incident of the taxi cab number…He went into the room where Ramanujan was lying. Hardy, always inept about introducing a conversation, said, probably without a greeting, and certainly as his first remark: "I thought the number of my taxi cab was 1729. It seemed to me rather a dull number." To which Ramanujan replied: "No, Hardy! No, Hardy! It is a very interesting number. It is the smallest number expressible as the sum of two cubes in two different ways."
>
> C. P. Snow (see page 37 of [18])

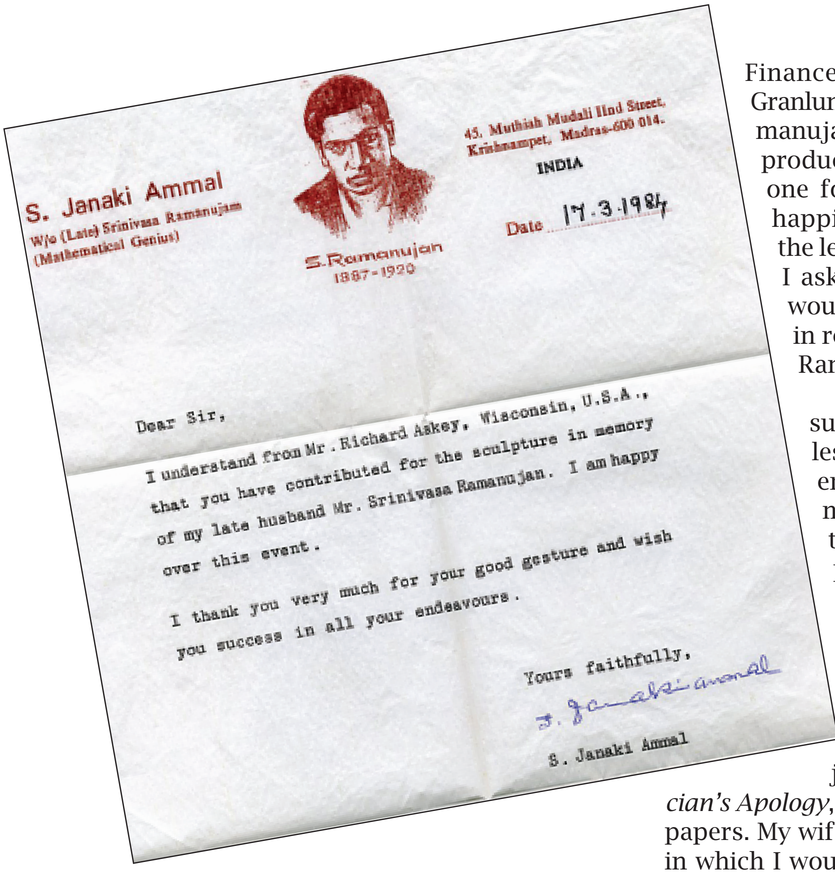
Indeed, we have

$$1729 = 1^3 + 12^3 = 10^3 + 9^3.$$

In the spring of 1919, Ramanujan returned to south India where he spent the last year of his life seeking health care and a forgiving climate. His health declined over the course of the following year, and he died on April 26, 1920, in Madras, with Janaki by his side. He was thirty-two years old.

My Pilgrimage

I first heard the story of Ramanujan when I was a reticent teenager obsessed with bicycle racing. It was a beautiful spring day in 1984, and my mind was on an important bicycle race in Washington D.C. when a letter adorned with Indian stamps arrived. The letter was dated 17-3-1984, and it was carefully typewritten on delicate rice paper. My father, Takashi Ono, a number theorist at Johns Hopkins University, was deeply moved by the letter which read [22]:



S. Janaki Ammal
W/o (Late) Srinivasa Ramanujam
(Mathematical Genius)

S.Ramanujan
1887-1920

45, Muthiah Mudali IInd Street,
Krishnampet, Madras-600 014.
INDIA

Date 17.3.1984

Dear Sir,

I understand from Mr. Richard Askey, Wisconsin, U.S.A., that you have contributed for the sculpture in memory of my late husband Mr. Srinivasa Ramanujan. I am happy over this event.

I thank you very much for your good gesture and wish you success in all your endeavours.

Yours faithfully,

S. Janaki ammal

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Dear Sir,

I understand from Mr. Richard Askey, Wisconsin, U.S.A., that you have contributed for the sculpture in memory of my late husband Mr. Srinivasa Ramanujan. I am happy over this event.

I thank you very much for your good gesture and wish you success in all your endeavours.

Yours faithfully,

Signed S. Janaki Ammal

My father explained that Dick Askey, a mathematician at the University of Wisconsin at Madison, had organized an effort, on behalf of the mathematicians of the world, to commission a sculpture of Ramanujan. This initiative was in response to an interview[4] with Janaki Ammal, Ramanujan's widow. She lamented,

> They said years ago a statue would be erected in honor of my husband. Where is the statue?

Financed by Askey's efforts, artist Paul Granlund rendered a sculpture based on Ramanujan's 1919 passport photo, and he produced eleven bronze casts, including one for Ramanujan's widow. My father happily contributed US$25, and hence the letter. Upon hearing this explanation, I asked, "Who was Ramanujan?" "Why would you give $25 expecting nothing in return?" That was when I first heard Ramanujan's story.

At the time, I had no plan of pursuing a career in mathematics, much less one involving Ramanujan's mathematics. As it was, the romantic tale made a lasting impression, and, thanks to my choice of career and the passage of time, has become one of my favorite stories.

Seven days ago I eagerly boarded a flight from Madison beginning my pilgrimage to Kumbakonam. In anticipation, I reread Kanigel's popular biography of Ramanujan [20] and Hardy's *A Mathematician's Apology*, among countless other articles and papers. My wife Erika gave me a beautiful journal in which I would go on to record pages of notes. Despite these preparations, I was unsettled. The long flights amplified these feelings. What was I looking for? After all, I did not expect to find a lost notebook, or acquire divine inspiration allowing me to prove famous open conjectures. I struggled with this question, and I ultimately decided that I should not ask it. I was content with the idea of simply paying homage to a great mathematician, one whose legend and work had become intertwined with the fabric of my life.

Despite my resolution, I was still bothered by two quotes from Hardy's 1936 Harvard tercentenary lectures on Ramanujan. He asserted (see page 4 of [17]),

> I am sure that Ramanujan was no mystic and that religion, except in a strictly material sense, played no important part in his life.

Could this be true? He also proclaimed (see page 5 of [17]),

> There is quite enough about Ramanujan that is difficult to understand, and we have no need to go out of our way to manufacture mystery.

Is it possible to rationally explain the legend of Ramanujan?

I arrived in Chennai at 8:45 a.m. on December 19, 2005, on a flight from Mumbai. The effects of several days of heavy rain were inescapable. South

[4] *From the article "Where is the statue?" in the June 21, 1981, issue of the* Hindu.

The highway from Chennai to Kumbakonam.

India was devastated by severe flooding. How would these conditions impact the 170-mile drive from Chennai to Kumbakonam that was scheduled for the afternoon?

I was shuttled across town to a local hotel where many of the invited speakers and their guests had gathered. There I enjoyed a quick lunch and a refreshing hot shower. Around 1:30 p.m. we departed for Kumbakonam in a minivan kindly provided by SASTRA University. The other mathematicians on board were Krishnaswami Alladi, Alexander Berkovich, Manjul Bhargava, Mira Bhargava (Manjul's mother), and Evgeny Mukhin.

The first hour of our journey was uneventful. In steady rain, we barely poked along in Chennai traffic snarled by auto-rickshaws, bicycles, livestock, and masses of people (many without footwear). Then out of the blue we found ourselves on India's celebrated national highway. Begun in 1991, the national highway program is a component in India's plan to advance its economy by improving infrastructure. The highway is distinctly Indian. Goats and cows appear at regular intervals, and people cross lanes of traffic on foot without fear. Imagine cows feeding on the grass on the median of a divided highway! Our speed rarely exceeded 45 miles per hour. The section of highway was quite short (perhaps 30 miles), and the balance of the route covered brutally rough roads. Some sections were so savage that we literally bobbed from rut to rut. I did my best to enjoy the sight of the beautiful lush green rice paddies and sugar cane fields as we bounced down the flood-ravaged road. Needless to say, the Sterling Resort, a rustic Indian-style hotel, was a welcome sight when we arrived at 9:00 p.m. The warm hotel staff draped lovely garlands around our necks and imprinted red *tilaks* on our foreheads. The glasses of rose water and foot massages which followed were perfect elixirs for such a grueling ride.

The next morning, after an exquisite breakfast of masala dosa, one of my favorite south Indian dishes, we boarded the minivan for the short drive to SASTRA University, the site of the International Conference on Number Theory and Mathematical Physics and home of the Srinivasa Ramanujan Centre. The day began with the awarding of the first SASTRA Ramanujan Prize, a prestigious international award recognizing research by young mathematicians (under the age of 32) working in areas influenced by Ramanujan. Arabinda Mitra, the executive director of the Indo-U.S. Science and Technology Forum, and Krishnaswami Alladi, the chair of the prize committee, jointly awarded Manjul Bhargava (Princeton University) and Kannan Soundararajan (University of Michigan) the prize for their respective works in number theory. The dazzling ceremony included the lighting of a stunning brass lamp, traditional Indian songs, and a passionate speech by Mitra announcing new scientific Indo-U.S. ventures. The majestic ceremony was a fitting amalgamation of Indian tradition with promising visions of the future. The spectacle was breathtaking: two young stars lauded in the name of Ramanujan in his hometown.

After a full slate of lectures, we were driven to two sacred sites: Ramanujan's childhood home and Sarangapani Temple. We first visited Ramanujan's home on Sarangapani Sannidhi Street. The one-story stucco house, which sits inconspicuously among a row of shops, is a source of national pride. In 2003, Abdul Kalam, the president of India, named it the "House of Ramanujan", and he dedicated it as a national museum.

The house does not possess any striking features. In the front there is a small porch, one of Ramanujan's favorite places to do mathematics. We took many photos of the porch, and we tried to imagine the sight of Ramanujan calculating power series there as a young boy. I spent the next half hour pacing through the tiny house which consists

Arabinda Mitra (Indo-U.S. Forum Chair), Kannan Soundararajan, Manjul Bhargava, and Krishnaswami Alladi after the prize ceremony.

Manjul Bhargava, left, and Ken Ono in front of Ramanujan's house.

of two rooms and a kitchen. The very small bedroom is immediately on your left as you enter through the front door, and its only distinguishing features are a window facing the street, and an old-fashioned bed occupying nearly half of the floor space. The exhibits in the museum are modestly displayed in the main room, and they include a bust of Ramanujan decorated with garlands. There was a beautiful *kolam* in front of the bust, an intricate floral-like symmetric design on the floor fashioned out of rice flour. These designs are replaced by careful hands daily, and they are meant to distract one's attention from beautiful objects thereby minimizing *dhrishti*, the effect of jealous eyes. Behind Ramanujan's house there is a tiny courtyard with an old well.

Two blocks away, the Sarangapani Temple towers over Ramanujan's neighborhood. There Ramanujan and his family regularly offered prayers to the Hindu god Lord Vishnu. There are accounts of Ramanujan working on mathematics in its great halls.

> Here, to the sheltered columned coolness, Ramanujan would come. Here, away from the family, protected from the high hot sun outside, he would sometimes fall asleep in the middle of the day, his notebook, with its pages of mathematical scrawl, tucked beneath his arm, the stone slabs of the floor around him blanketed with equations inscribed in chalk.
>
> R. Kanigel (see pages 29-30 of [20])

The brilliant orange hue of the sun's rays encircling the colossal structure, like the corona of the sun, beckoned us from the porch of Ramanujan's house. The temple, built mostly between the 13th and 17th centuries, is a twelve-storied superstructure constructed from stone brought from the north by elephants. The temple is tetragonal, and its outer walls are completely covered with colorful ornate carvings depicting countless Hindu legends.

After we passed beneath the *gopuram*, the temple gate, dozens of bats circled above us against the dim lit sky. A few steps away, there were several cows chomping on hay. The interior of the temple is a stunning labyrinth of sculptures, stone columns, brass walls, flickering lights and candles, and brass pillars. The walls are completely covered with ornate metalwork and stone carvings. Honoring Hindu tradition, we stepped barefoot over the stone floor in a clockwise direction. Along our path we passed dozens of *kolam* floor designs. The air was warm and muggy, and heavy with the scent of incense. The main central shrine is a monolith resembling a chariot drawn by horses and elephants.

Beyond the monolith lies the inner sanctum, protected by a pair of ancient bulky wooden doors covered with bells. The inner sanctum, bursting with silver and bronze vessels, is the bronze-walled resting place of Lord Vishnu. Krishnaswami Alladi and his wife, Mathura, called us into the inner sanctum and made offerings of coconuts and vegetables to Lord Vishnu via the Hindu priests. I understood that Alladi arranged for us to be blessed in an impassioned *pooja*, or prayer ceremony.

As we made our way out of the temple, I came upon a small set of steps that led to a stone cubbyhole containing the statue of a Hindu god flanked

Sarangapani Temple.

Portrait of a young Ramanujan at Town High School.

by melted candles. This nook took my breath away; its stone walls were covered by numbers scrawled in charcoal. I was so pleased; how appropriate for Ramanujan's temple to be covered with numbers! Sound's father, Soundararajan Kannan, explained that it is not unusual for Hindus to etch important numbers when making offerings. Some numbers were birthdates, while others appeared to be telephone numbers. As I surveyed the numbers, I excitedly searched for 1729, the taxi cab number. I never spotted it, but to my amazement I found

$$2719$$

prominently etched at eye level. For me this number plays a special role in the lore of Ramanujan, not only as a permutation of the digits of 1729, but for its connection to his work on quadratic forms. In 1997 Sound and I proved [21], assuming the Generalized Riemann Hypothesis, that 2719 is the largest odd number not represented by Ramanujan's ternary quadratic form

$$x^2 + y^2 + 10z^2.$$

I was delighted to see it near where Ramanujan worked a century ago.

The next day provided another full slate of talks. My student Karl gave a superb talk on his research on the Andrews-Garvan-Dyson "crank" and its role in describing Ramanujan's partition congruences. I gave my lecture on mock theta functions and Maass forms. Later we boarded the minivan for further sightseeing. We visited Town High School, where Ramanujan excelled before his addiction to mathematics, and Government College, the first college to flunk Ramanujan.

Just before I had left the U.S., I spoke with Bruce Berndt, a professor at the University of Illinois and acclaimed Ramanujan expert. From him I learned that I could see the original copy of Carr's book, the one that Hardy said (see page 3 of [17]) "awakened his [Ramanujan's] genius". When Berndt last visited Kumbakonam, the book was on display in the library at Government College. After this conversation, I imagined flipping through the pages (if allowed) for evidence of Ramanujan's handiwork. Perhaps I would discover elegant formulas delicately noted in the margin of the book.

Shortly after we set foot on campus, I heard the devastating news. The book was lost. My disappointment quickly turned to anger. How does one lose such a prominent artifact, one which is central to the story of Ramanujan? As I write this, I now prefer to think that the book is not lost, but borrowed by a connoisseur who adores it, much like an art collector might cherish masterpieces bought on the black market. When it reappears, I hope it finds its way to the House of Ramanujan.

After the short visit to Government College, we made our way to Town High School, site of Ramanujan's first academic successes. We arrived after classes had ended for the day. The school is an impressive two-story building with arched balconies and a lush tropical courtyard. My spirits were quickly lifted by A. Ramamoorthy and S. Krishnamurthy, two of the school's teachers. They kindly gave us an entertaining tour of campus, which included a stop in *Ramanujam Hall*,[5] a cavernous room dedicated to the memory of Ramanujan. The teachers also proudly displayed copies of awards that Ramanujan won as a top student. I was deeply moved by the pride with which they shared their campus and revelled in the story of Ramanujan. Their passion confirms that Ramanujan's status as a national hero endures today.

Near the end of our visit, Ramamoorthy revealed that he teaches English, and as a student was never very good at math. He timidly asked whether I could explain any of Ramanujan's work to him, and based on his facial expression it was clear he expected a negative answer. I was thrilled by the challenge, and I found a chalkboard and explained Ramanujan's partition congruences. A *partition* of an integer n is any nonincreasing sequence of positive integers that sum to n, and the partition function $p(n)$ counts the number of partitions of n. There are five partitions of four, namely

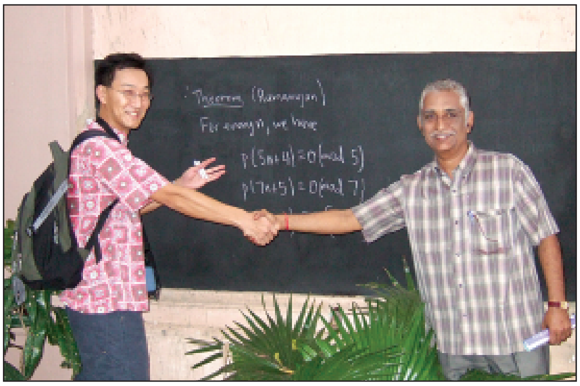
$$4, \ \ 3+1, \ \ 2+2, \ \ 2+1+1, \ \ 1+1+1+1,$$

and so $p(4) = 5$. The simplest examples of Ramanujan's congruences assert that

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5}, \\ p(7n+5) &\equiv 0 \pmod{7}, \\ p(11n+6) &\equiv 0 \pmod{11} \end{aligned}$$

for every integer n. As is common in number theory, the problems and theorems are often easy to explain (but hard to prove). My new friends were delighted by the simplicity of the congruences,

[5] *The teachers explained that Ramanujan can be spelled Ramanujam due to transliteration.*



K. Ono, left, and A. Ramamoorthy at Town High School.

and they promised to share them with the students the next day, Ramanujan's birthday.

The conference also concluded the next day. Manjul Bhargava closed the conference by delivering the Ramanujan Commemorative Lecture, a captivating talk on his recent work with Jonathan Hanke (Duke University). His topic came as a surprise; I had been expecting to hear him lecture on the Cohen-Lenstra heuristics, and generalizations of Gauss' composition laws. Instead, he announced new theorems about integral quadratic forms.

The study of integral quadratic forms, which dates to classic works of Jacobi, Lagrange, Fermat, and Gauss, plays an important role in the history of number theory. Indeed, Lagrange's Theorem that every positive integer is a sum of four squares is a classic result that number theory students learn early on. Revisiting earlier work of John H. Conway (Princeton University) and William Schneeberger, Manjul and Hanke have proven delightful results establishing finite tests for determining whether a quadratic form represents all positive integers. Consequences of their work are easy to state. For instance, they show that a positive-definite integral quadratic form represents all positive integers if and only if it represents the integers

$$1, 2, 3, 5, 6, 7, 10, 13, 14, 15, 17, 19, 21,$$
$$22, 23, 26, 29, 30, 31, 34, 35, 37, 42, 58,$$
$$93, 110, 145, 203, \text{ and } 290.$$

As a corollary, they determine the complete list[6] of all the positive-definite integral quadratic forms in four variables that represent all positive integers. This resolved a problem first studied by Ramanujan in his classic 1916 paper [24] on quadratic forms.

Bhargava has obtained even more general results. He shows that for every subset S of the positive integers, there is a unique minimal finite subset of integers, say T, with the property that such a form represents all the integers in S if and only if it represents the integers in T. Manjul concluded his lecture with a discussion of the following open problem: Determine T when S is the set of positive odd numbers. This problem is open due to deep questions in analytic number theory, most prominently the ineffectivity of Siegel's lower bound for class numbers, and to a lesser extent, a case of the Ramanujan-Petersson Conjectures. The celebrated effective solution of Gauss' general class number problem due to the work of Goldfeld, Gross, and Zagier, which provides an effective lower bound for class numbers, unfortunately falls short for this problem.

Manjul noted that Ramanujan, in his 1916 paper [24], had already anticipated these difficulties when he proclaimed (see page 14 of [24]):

> ...the even numbers which are not of the form $x^2 + y^2 + 10z^2$ are the numbers
>
> $4^{\lambda}(16\mu + 6)$,
>
> while the odd numbers that are not of that form, viz.,
>
> $$3, 7, 21, 31, 33, 43, 67, 79, 87, 133,$$
> $$217, 219, 223, 253, 307, 391, \ldots$$
>
> do not seem to obey any simple law.

In the 1980s Duke and Schulze-Pillot [14, 15] used deep results of Iwaniec [19] on the Ramanujan-Petersson Conjecture for half-integral weight modular forms to prove that there are *only finitely many* odd numbers that are not this form, guaranteeing that there is a "simple law" that they obey. However, the catch is that the proof is ineffective, meaning that it cannot be used to deduce the finite list of rogue exceptions. This sort of predicament explains the nature of Manjul's open problem.

On one of his final slides, Manjul recalled my result with Sound which brightened the picture:

Assuming the Generalized Riemann Hypothesis, the only odd numbers not of this form are

$$3, 7, 21, 31, 33, 43, 67, 79, 87, 133,$$
$$217, 219, 223, 253, 307, 391, 679, \text{ and } \mathbf{2719}.$$

This was a poetic conclusion to my pilgrimage: the number

2719

echoing from a small cubbyhole in the great hall of Ramanujan's temple.

Ramanujan's Mathematical Legacy

To properly appreciate the legend of Ramanujan, it is important to assess his legacy to mathematics. For this task, we recall the thoughts (see page

[6] *Lagrange's form $w^2 + x^2 + y^2 + z^2$ is one of them.*

xxxvi of [16]) Hardy recorded shortly after Ramanujan's death in 1920:

> Opinions may differ about the importance of Ramanujan's work, the kind of standard by which it should be judged, and the influence which it is likely to have on mathematics of the future. …He would probably have been a greater mathematician if he could have been caught and tamed a little in his youth. On the other hand he would have been less of a Ramanujan, and more of a European professor, and the loss might have been greater than the gain….

Sixteen years later, on the occasion of Harvard's tercentenary, Hardy revisited this quote, and he retracted (see page 7 of [17]) the last sentence as "ridiculous sentimentalism".

In light of what we know now, perhaps we should revisit this decision. With the passage of time, it should be much simpler to assess Ramanujan's legacy. Indeed, we enjoy the benefit of reflecting on eighty-five years of progress in number theory. However, the task is complicated at many levels. It would be unfair to assess his legacy based on his published papers alone. The bulk of his work is contained in his notebooks. This is underscored by the fact that the project of editing the notebooks remains unfinished, despite the tireless efforts of Bruce Berndt over the last thirty years, adding to the accumulated effort of earlier mathematicians such as G. H. Hardy, G. N. Watson, B. M. Wilson, and R. A. Rankin. The task is further complicated by the fact that modern number theory bears little resemblance to Ramanujan's work. It is safe to say that most number theorists, unfamiliar with his notebooks, would find it difficult to appreciate the pages of congruences, evaluations, and identities, of strangely named functions, as they are presented in the notebooks. To top it off, these results were typically recorded without context, and often without any indication of proof. Our task would be far simpler had Ramanujan struck out and developed new theories whose fundamental results are now bricks in the foundation of modern number theory. But then he would have been "less of a Ramanujan".

Despite these challenges, it is not difficult to paint a picture that reveals the breadth and depth of Ramanujan's legacy. Instead of concentrating on examples of elegant identities and formulas, which is already well done in many accounts by mathematicians such as Berndt and Hardy (for example, see [5, 6, 7, 8, 9, 17, 25]), we adopt a wider perspective that illustrates Ramanujan's influence on modern number theory.

Number theory has undergone a tremendous evolution since Ramanujan's death. The subject is now dominated by the arithmetic and analytic theory of automorphic and modular forms, the study of Diophantine questions under the rubric of arithmetical algebraic geometry, and the emergence of computational number theory and its applications. These subjects boast many of the most celebrated achievements of twentieth century mathematics. Examples include: Deligne's proof of the Weil Conjectures, the effective solution of Gauss' general Class Number Problem (by Goldfeld, Gross, and Zagier), Wiles' proof of Fermat's Last Theorem, and Borcherds' work on the infinite product expansions of automorphic forms. At face value, Ramanujan's work pales in comparison. However, in making this comparison we have missed an important dimension to his genius: his work makes contact with all of these notable achievements in some beautiful way. Ramanujan was a great anticipator; his work provided examples of deeper structures and suggested important questions that now permeate the landscape of modern number theory.

To illustrate this, consider Ramanujan's work on the single function

$$\begin{aligned}(1)\qquad \Delta(z) &= \sum_{n=1}^{\infty} \tau(n)q^n := q\prod_{n=1}^{\infty}(1-q^n)^{24}\\ &= q - 24q^2 + 252q^3 - 1472q^4 + \cdots,\end{aligned}$$

where $q := e^{2\pi i z}$ and z is a complex number with $\mathrm{Im}(z) > 0$. Viewing this function as a formal power series in q, one would not suspect its important role. This function is a prototypical *modular form*, one of weight 12. As a function on the upper half of the complex plane, this essentially means that

$$\Delta\left(\frac{az+b}{cz+d}\right) = (cz+d)^{12}\Delta(z)$$

for every matrix $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$. Ramanujan was enraptured by its coefficients $\tau(n)$, the values of the so-called tau-function.

Although nothing about their definition suggests such properties, Ramanujan observed and conjectured (see page 153 of [25]) that

$$\tau(nm) = \tau(n)\tau(m)$$

for every pair of coprime positive integers n and m, and that

$$\tau(p)\tau(p^s) = \tau(p^{s+1}) + p^{11}\tau(p^{s-1})$$

for primes p and positive integers s. Although Mordell would prove these conjectures, those with knowledge of modular forms will recognize them as by-products of a grand theory that would be developed in the 1930s by E. Hecke. The modern theory of automorphic and modular forms and their L-functions, which dominates much of modern number theory, is a descendant of Hecke's theory.

In addition to studying their multiplicative properties, Ramanujan studied the size of the numbers

$\tau(n)$. For primes p he conjectured (see pages 153-154 of [25]), but could not prove, that

$$|\tau(p)| \leq 2p^{\frac{11}{2}}. \tag{2}$$

This speculation is the first example of a family of conjectures now referred to as the Ramanujan-Petersson Conjectures, among the deepest problems in the analytic theory of automorphic and modular forms. This conjectured bound was triumphantly confirmed [13] by Deligne as a deep[7] corollary of his proof of the Weil Conjectures, work that earned him the Fields Medal in 1978. Although it would be ridiculous to say that Ramanujan anticipated the Weil Conjectures, which includes the Riemann hypothesis for varieties over finite fields, he correctly anticipated the depth and importance of optimally bounding coefficients of modular forms, the content of the Ramanujan-Petersson Conjectures.

As another example of Ramanujan the anticipator, we reflect on the many congruences he proved for the tau-function, such as (see page 159 of [25]):

$$\tau(n) \equiv \sum_{d|n} d^{11} \pmod{691}. \tag{3}$$

Although this congruence is not difficult to prove using q-series identities, it provides another example of a deep theory. About thirty-five years ago, Serre [26] and Swinnerton-Dyer [28] wrote beautiful papers interpreting such congruences in terms of certain two dimensional ℓ-adic representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the absolute Galois group of the algebraic closure of $\mathbb{Q}$. At the time, Deligne had just proven that such representations encode the coefficients of certain modular forms as "traces of the images of Frobenius elements". Armed with this perspective, Serre and Swinnerton-Dyer interpreted Ramanujan's tau-congruences, such as (3), as the first nontrivial examples of certain "exceptional" representations. In the case of Ramanujan's $\Delta(z)$, for the prime $\ell = 691$, there is a (residual) Galois representation

$$\rho_{\Delta,691} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{Z}/691\mathbb{Z})$$

which, for primes $p \neq 691$, satisfies

$$\rho_{\Delta,691}(\mathrm{Frob}(p))) = \begin{pmatrix} 1 & * \\ 0 & p^{11} \end{pmatrix},$$

where $\mathrm{Frob}(p) \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ denotes the "Frobenius element at p". Congruence (3) then follows from Deligne's prescription, for primes $p \neq 691$, that

$$\mathrm{Tr}(\rho_{\Delta,691}(\mathrm{Frob}(p))) \equiv \tau(p) \pmod{691}.$$

This theory of modular ℓ-adic Galois representations, which provides Galois-theoretic interpretations of Ramanujan's tau-congruences, has subsequently flourished over the years, and famously is the "language" of Wiles' proof of Fermat's Last Theorem.

As one readily sees, Ramanujan's work on the tau-function anticipated deep theories long before their time. Similar remarks apply to much of Ramanujan's work. Over the last few years, largely due to work of Zagier and Zwegers [30, 31],[8] a clear picture has begun to emerge concerning the *mock theta functions*, the focus of Ramanujan's work while he was bedridden in his last year of life. These strange q-series, such as

$$f(q) := 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{(1+q)^2(1+q^2)^2 \cdots (1+q^n)^2}$$
$$= 1 + q - 2q^2 + 3q^3 - \cdots,$$

are related to *Maass forms*, a type of nonholomorphic modular form that would not be defined until the 1940s, twenty years after Ramanujan's death. Thanks to these new connections, several longstanding open problems about mock theta functions and partitions have recently been solved (for example, [11, 12]). Research in this direction is presently advancing at a rapid rate, and although the details have not yet been fully worked out, it should turn out that mock theta functions will also provide examples of automorphic infinite products. These products were introduced by Borcherds in his 1994 lecture at the International Congress of Mathematicians [10]. These products, combined with his work on Moonshine, earned Borcherds a Fields Medal in 1998.

In other areas of number theory, Ramanujan's legacy and genius stand out further in relief. He was a pioneer in probabilistic number theory, in the theory of partitions and q-series, and in the theory of quadratic forms, and together with Hardy he gave birth to the "circle method", a fundamental tool in analytic number theory (for example, see [5, 6, 7, 8, 9, 16, 25]). His work in these subjects, combined with the deep theories he anticipated, paints a breathtaking picture of his mathematical legacy.

As a final (crude) measure of Ramanujan's legacy, simply consider the massive list of mathematical entities that bear his name:

- The Dougall-Ramanujan identity
- The Landau-Ramanujan constant
- Ramanujan's theta-function
- Ramanujan's class invariants g_n and G_n
- Ramanujan's ${}_1\psi_1$ identity
- Ramanujan's τ-function

[7] *Earlier works by Eichler, Ihara, Sato, and Shimura play an important role in reducing (2) to a consequence of the Weil Conjectures.*

[8] *This research comprises Zwegers' Ph.D. thesis written under the direction of Don Zagier.*

- Ramanujan's continued fraction
- Ramanujan graphs
- Ramanujan's mock theta functions
- The Ramanujan-Nagell equation
- The Ramanujan-Petersson Conjectures
- Ramanujan sums
- Ramanujan's theta-operator
- The Rogers-Ramanujan identities
- among many others…

If Hardy knew what we know now, perhaps he would again alter his 1920 quote. Rather than dismissing the last sentence as "ridiculous sentimentalism", perhaps he would agree that it rings true now more than it originally had at the time of Ramanujan's death.

Reflections

I am compelled to return to the quotes by Hardy which prompted me to consider whether it is possible that religion was not an important part of Ramanujan's life, and whether one can rationally explain the legend of Ramanujan's mathematics.

I certainly cannot resolve the question of whether religion was an important part of his life. Obviously, I also cannot truly speculate on whether he believed his research was divine in origin. That would be romantic fiction. However, based on my experiences, particularly my visit to Sarangapani Temple, it is difficult to imagine that religion did not play some role. From a western perspective, it is hard to overstate the importance and relevance of Hindu beliefs on all aspects of daily life in Kumbakonam. Hinduism permeates daily life. After all, Kumbakonam is a holy city, one where ninety percent of its citizens today are observant Hindus, a fact that was certainly true in Ramanujan's day. It is also difficult to ignore the well-documented fact that Komalatammal, Ramanujan's mother, was deeply religious and that his voyage to England was dependent on her dream from the goddess Namagiri. Therefore whether Ramanujan was deeply religious or not, it is certainly true that everything about him and his world view was heavily influenced by religion.

For me, there is a poetic resolution to the question of whether one can rationally explain the legend of Ramanujan: this true story is one of magic. Ramanujan was an untrained mathematician, toiling largely in isolation, whose work was born entirely out of imagination. He was a pioneer and a self-taught anticipator of great mathematics, and this is indeed magical. After all, great mathematics is magic, something we can understand but whose inspiration we cannot comprehend. Ramanujan was a gift to the world of mathematics.

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Note: All photographs used in this article were taken by the author, Ken Ono.

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About the Cover

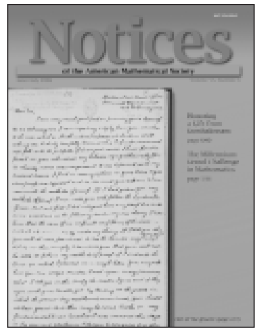
Out of the Groove

Madras Port Trust Office
Accounts Department.
27th February 1913.

Dear Sir,

I am very much gratified on perusing your letter of the 8th February 1913. I was expecting a reply from you similar to the one which a Mathematics Professor at London wrote asking me to study carefully Bromwich's Infinite Series and not fall into the pitfall of divergent series. I have found a friend in you who views my labours sympathetically. This is already some encouragement to me to proceed with an onward course. I find in many a place in your letter rigourous proofs are required and so on and you ask me to communicate the methods of proof. If I had given you my methods of proof I am sure you will follow the London Professor. But as a fact I did not give him any proof but made some assertions as the following under my new theory. I told him that the sum of an infinite number of terms in the series $1 + 2 + 3 + 4 + \cdots = -1/12$ *under my theory. If I tell you this you will at once point out to me the lunatic asylum as my goal. I dilate on this simply to convince you that you will not be able to follow my methods of proof if I indicate the lines on which I proceed in a single letter. You may ask how you can accept results based upon wrong premises. What I tell you is this. Verify the results I give and if they agree with your results, got by treading on the groove in which the present day mathematicians move, you should at least grant that there may be some truths in my fundamental basis. So what I now want at this stage is for eminent professors like you to recognize that there is some worth in me. I am already a half starving man. To preserve my brain I want food and this is now my first consideration. Any sympathetic letter from you will be helpful to me here to get a scholarship either from the University or from Government.*

With respect to the mathematics portion of your letter…



This is the beginning of the second letter from Ramanujan to G. H. Hardy. The first, one of the most famous of all documents in the history of mathematics, had been written on January 16, and Hardy had replied from Trinity College, Cambridge, on February 8. The beginning of the first letter seems unfortunately to have disappeared, although its content has been preserved. Hardy commented in a note written July 23, 1940, "I have looked in all likely places, and can find no trace of the missing pages of the first letter, so I think we must assume that it is lost. This is very natural since it was circulated to quite a number of people interested in Ramanujan's case."

Both letters as well as other relevant items can be read in Bruce Berndt's account in *Ramanujan—Letters and Commentary*, published by the AMS. The idea of making this cover came from Ken Ono's article in this issue (pp. 640–51).

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—Bill Casselman, Graphics Editor
(notices-covers@ams.org)