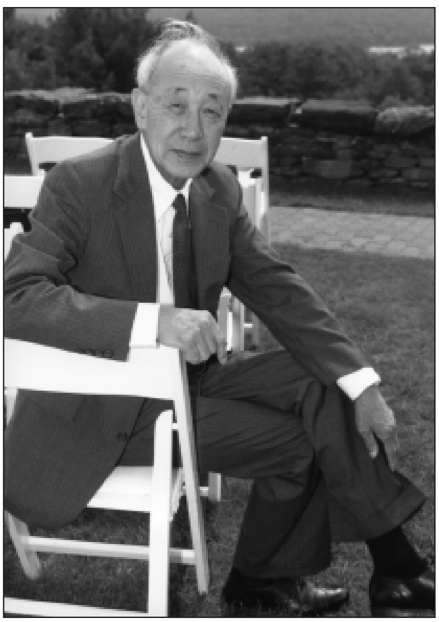
Teruhisa Matsusaka (1926–2006)

János Kollár



Teruhisa Matsusaka

From his arrival at the Unversity of Chicago in 1954 until his retirement from Brandeis University in 1994, Matsusaka has been a key figure of American algebraic geometry. He was a quiet mathematician to whom algebraic geometry was a personal friend whose company one can best appreciate away from the rush of the academic life. Instead of going to conferences or working with others, he most enjoyed sitting in his fishing boat and thinking about mathematics. Yet for those who knew him well, his love of the subject and his devotion to the deep understanding of important problems was infectious.

Matsusaka received his Ph.D. in 1952 at Kyoto University, but the person with the greatest influence on his research career was André Weil. During the difficult years after the Second World War, Matsusaka worked on several problems connected with Weil's *Foundations of Algebraic Geometry*. This led to a correspondence and eventually Weil invited Matsusaka to the University of Chicago (1954–57) where they became life-long friends.

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After three years at Northwestern Unversity and a year at the Institute for Advanced Study, Princeton, he went to Brandeis University in 1961 where he stayed until 1994, helping to build the department to its current prominence. Matsusaka was invited to address the Edinburgh International Congress of Mathematicians in 1958, and he was elected to the American Academy of Arts and Sciences in 1966.

Most of the early works of Matsusaka are devoted to extending basic theorems of complex algebraic geometry to arbitrary fields, following the direction established by Weil. These theorems carry the name of the original authors who proved them over the complex numbers, and very few of the young generation know that the existence of the Albanese and Picard varieties [Mat52], the projectivity of Abelian varieties [Mat53], the Lefschetz theorem computing the Picard variety of a hyperplane section [Mat54], and the Torelli theorem that the polarized Jacobian determines a smooth curve [Mat58] are all due to Matsusaka in the general case.

It is also in these early years that he proved a characterization of Jacobians, now known as the Matsusaka criterion [Mat59]:

A principally polarized Abelian variety of dimension g containing a curve $(C \subset A, \Theta)$ is the Jacobian of C if and only if $(g-1)! \cdot C$ is numerically equivalent to the self-intersection Θ^{g-1}.

After these results Matsusaka turned his attention to the moduli problem of algebraic varieties, and he devoted the rest of his career to this topic. Nowadays moduli problems are built into the basics of algebraic geometry, but in the 1950s and

1960s the existence and nature of moduli spaces was a major open question, where even the basic definitions were unsettled.

It was realized early on that the set of isomorphism classes of varieties of a given type does not carry any reasonable algebraic structure. Some of the worst examples are given by Abelian and K3 surfaces.

Matsusaka devoted thirty years of work to proving that the right objects to consider are *polarized* varieties, that is, pairs (X, H) where X is a projective variety and H is an ample divisor class.

Two major preliminary questions need to be settled before one can start building a sensible moduli space, and both were solved by Matsusaka: is the moduli problem *separated* and *bounded*?

André Weil and Ryoko Matsusaka, 1956. ***Photo by Teruhisa Matsusaka.***

The separatedness problem concerns uniqueness of limits. Assume that (X_t, H_t) depend continuously or algebraically on t where $t \neq 0$. Is it true that there is at most one pair (X_0, H_0) that can be viewed as the limit of this family?

It is easy to write down examples where this completely fails if we allow X_0 to be singular. It is harder to come up with smooth counterexamples, but $\mathbb{P}^1$-bundles over curves give many. The basic paper of Matsusaka and Mumford [MM64] shows that these are the only obstructions: the limit is unique as long as X_0 is smooth and not birational to a $\mathbb{P}^1$-bundle.

The boundedness problem asks about the totality of all pairs (X, H). Maybe there are too many of them to be parametrized by the points of a single algebraic variety? This happens already for curves: we need infinitely many algebraic varieties to parametrize all polarized curves (C, H), but we need only one variety if we fix the genus of C and the degree of H. Equivalently, we can fix the Hilbert polynomial $\chi(C, \mathcal{O}_C(tH)) = t \cdot (\text{degree of } H) + 1 - g(C)$.

Similarly, in higher dimensions we need to fix the Hilbert polynomial

$$Hilb_{(X,H)}(t) := \chi(X, \mathcal{O}_X(tH))$$

first. It turns out that one can restate boundedness in the following form:

For every polynomial $p(t)$ find a constant $c = c(p(t))$ such that if (X, H) is a polarized pair with $p(t) = Hilb_{(X,H)}(t)$ then $|cH|$ is very ample.

Matsusaka's best known result [Mat72], dubbed *Matsusaka's Big Theorem* by Lieberman and Mumford [LM75], gives the positive answer to this question when X is smooth and the characteristic is 0.

Later Matsusaka showed that the answer is also positive if X is smooth, $\dim X \leq 3$ [Mat81], and the characteristic is arbitrary, or when X has rational singularities and the characteristic is 0 [Mat86]. (The more general cases of smooth varieties in positive characteristic and normal varieties in characteristic 0 are still unknown.)

Once boundedness holds, one can parametrize all pairs (X, H) with a given Hilbert polynomial by an open subset of the Chow variety (or Hilbert scheme) of a fixed projective space $\mathbb{P}^N$. The remaining problem is that we obtain every (X, H) many times, since the embedding of X into $\mathbb{P}^N$ is unique only up to an automorphism of $\mathbb{P}^N$. This leads us to the question: when is the quotient of an algebraic variety by a group also an algebraic variety? In some cases the answer is given by Mumford's geometric invariant theory or by Artin's theory of algebraic spaces, but a truly satisfactory general answer is still lacking. Matsusaka's main contribution to this area is the theory of Q-varieties [Mat64]. This is the first clear instance in the literature where the currently popular "stacky viewpoint" appears in algebraic geometry. In both approaches the starting point is the dictum that one should not worry about the existence of the quotient. Instead, one should work out a theory of the more general quotient objects. Afterwards we may prove that the quotient also exists in the classical sense, but at the end this may turn out to be unimportant.

Following the theory of Weil, Matsusaka always worked with varieties, so his theory is about quotients of varieties by equivalence relations. By the time his book appeared in 1964, schemes took over algebraic geometry, and the current theory of stacks preserved only the basic philosophy of his approach.

Matsusaka's last published paper [Mat91] strengthens his *Big Theorem* by proving that only the two leading terms of the Hilbert polynomial $Hilb_{(X,H)}(t)$ matter. Namely, if $\dim X = n$ and H is ample (or just nef and big) then there is an effectively computable function $C(x, y, z)$ such that

$$\left| h^0(X, \mathcal{O}_X(tH)) - \frac{H^n}{n!} t^n + \frac{K_X \cdot H^{n-1}}{2(n-1)!} t^{n-1} \right| \leq C(H^n, K_X \cdot H^{n-1}, n) t^{n-2},$$

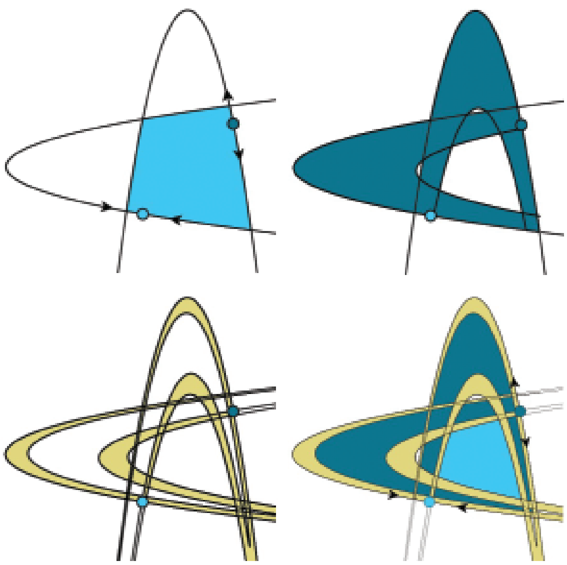
About the Cover

A Hénon horseshoe

This month's cover was suggested by Ruelle's article on strange attractors. It portrays an approximation to the non-wandering set Ω, as well as a portion of the homoclinic tangle, for the Hénon map

$$f: (x, y) \to (y, 1 - ay^2 + bx)$$

with $a = 6$, $b = 0.9$. For these values, the Hénon map does not possess an attractor, but instead offers an instance of one of Smale's horseshoes, and that is what the cover more or less illustrates. The logic is exhibited in more detail by the following sequence of pictures:



We begin with a region R bounded by parts of the stable manifold of one fixed point and the unstable manifold of the other. Any point outside R is taken off to infinity by iterates of either f or f^{-1}, so Ω lies inside it.The next pictures show $f(R)$ and $f^{-1}(R)$, then $f^2(R)$ and $f^{-2}(R)$. Each intersection $f^n(R) \cap f^{-n}(R)$ also contains Ω.

The symbolic dynamics of this example, the same as those of the horseshoe, are simple. As Ruelle mentions, the more complicated symbolic dynamics of the classical Hénon map with $a = 1.4$, $b = 0.3$ are specified by the *pruning front conjecture*. For more information, look at the AMS Feature Column for June 2006 (available at `http://www.ams.org/featurecolumn/`) and the book **Classical and Quantum Chaos** by Cvitanović and others, available at `http://ChaosBook.org`.

—Bill Casselman
Graphics Editor

and a similar inequality holds when we replace $h^0(X, \mathcal{O}_X(tH))$ by $\chi(X, \mathcal{O}_X(tH))$. Note that the Hirzebruch–Riemann–Roch theorem computes $\chi(X, \mathcal{O}_X(tH))$ exactly in terms of all the Chern classes of X, whereas the above result needs only the first Chern class $K_X = -c_1(X)$.

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