Book Review

Stalking the Riemann Hypothesis

Reviewed by John Friedlander

Stalking the Riemann Hypothesis
Dan Rockmore
Pantheon Books (2005)
$25.00, 304 pages, ISBN 0-375-42136-X

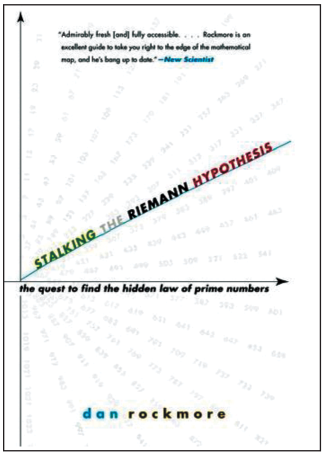
Now, thanks I guess to Andrew Wiles, the competition has just about disappeared. There is no Fermat conjecture to cloud the issue. The Riemann hypothesis reigns supreme as the mathematical goal which is to be used to grab the public's attention. How else to explain the sudden proliferation; this is at least the third[1] popular book on the subject to appear since 2003, the others being:

[SAB] K. Sabbagh, *The Riemann Hypothesis. The Greatest Unsolved Problem in Mathematics.*

[SAU] M. du Sautoy, *The Music of the Primes. Searching to Solve the Greatest Mystery in Mathematics.*

The reviewer has now read all three of these, in order, in each case having no dream at all that he would ever find occasion to read another popular book on this topic.

Each of these books is aimed at two audiences and, as such, faces somewhat of a dilemma. To quote Rockmore, "To distill years, even centuries, of scientific investigation for a broad and curious audience, while not raising the hackles of the experts in the field is something of an intellectual tightrope walk."



To a certain extent this is not as serious a problem for [SAB] who is not a mathematician. He exhibits less need to try to make the mathematics understandable. Although spending some of his time on this, he concentrates more heavily on telling amusing stories about various of the players in this drama, a part which I found rather enjoyable. Unfortunately, in the latter part of the book, well, to quote from Heath-Brown's Mathematical Review[2] of [SAB], "The human story is dominated by an account of Louis de Branges and his work on the Riemann hypothesis, to which the author devotes four chapters and an appendix. The reviewer found this rather depressing reading …" So did I.

The book by Rockmore is closer in spirit to the book [SAU] and, in the reviewer's mostly positive opinion, is of roughly equal merit. Either book is enjoyable, but perhaps most people would find reading both of them to be a little too much. Like

John Friedlander is University Professor of Mathematics at the University of Toronto. His email address is frdlndr@utsc.utoronto.ca.

[1] *I thank Allyn Jackson for drawing my attention to a fourth such book,* Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem*, by John Derbyshire.*

[2] *Apart from this one exception, all quotation marks refer to excerpts taken directly from the book under review.*

that book, it is written by a mathematician (although neither is an analytic number theorist; I wonder if any such would dare attempt this job). Hence, there is a genuine effort to make the mathematics understandable. Such an effort necessarily involves using, here, there, and everywhere, approximations to the truth and, given the audience, these approximations need to be much cruder than those we use in our survey articles or our colloquium lectures when we are addressing the well-known general mathematical audience. How well does Rockmore succeed in making his descriptions convey, to a nonmathematical audience, the idea of just what mathematics is about? I think he does a pretty good job of this although I am not at all certain that this is a question that can or should be answered by a mathematician.

Where would a book on the Riemann hypothesis begin? Would you believe the answer is Greenwich Village? Quickly, though, we dart back to Greece for, after all, at least for us eurocentrics, "This is number theory and the Pythagoreans are history's first number theorists." We begin with the primes as "the integral atoms", move on to the fundamental theorem of arithmetic, the sieve of Eratosthenes, and Euclid's proof of the infinitude of primes.

Soon, after some passages on how to count things asymptotically, we jump all the way forward to the "primal cartographers" Legendre and Gauss and their conjectures which would eventually become known as the prime number theorem. Here and at various places, other mathematical topics get briefly mentioned: non-Euclidean geometry, differential geometry, fast Fourier transform. Gauss is Gauss, after all.

The next chapter is entitled "Shoulders to stand on". Of course the quotation is due to Newton but, in this case, the shoulders are due to Euler. Back we go. Euler, when he was not being a "bridge builder" in Königsberg or any number of other things, was giving "a harmonious proof for Euclid". Specifically, he introduced, for many different purposes, the use of generating functions and, in particular, the one which was going to become the Riemann zeta-function. He developed a number of its most important properties, and he used it to study the primes and to give a proof that the sum of the reciprocals of the primes is divergent, a stronger version of Euclid's result.

Before we get to Riemann we need a couple of other shoulders, those of Dirichlet. Dirichlet took Euler's zeta-function and considered it as a function of a real variable, whereas Euler had been interested in it mainly at the integers. As a result, Dirichlet could consider limits. He also introduced L-functions which generalized the zeta-function and which allowed him, starting from Euler's ideas, to fashion a proof that every arithmetic progression of integers (apart from certain trivial counterexamples) contains infinitely many primes. Analytic number theory was born.

Next we come to the main event: "Riemann was waiting." After a brief description of his early years and his work on Riemannian geometry we come back to the primes. This chapter requires some harder work for the general audience. "The range of tools and techniques that Riemann would bring to bear on this problem make it seem as if he had been preparing all his life for this moment …These advances in the study of complex numbers, complex analysis and Fourier analysis are crucial to the Riemann hypothesis …" After several pages of preparation the audience is led (of course there are no precise statements) to the zeta-function as a function of a complex variable, the explicit formula which gives the prime-counting function in terms of the zeros of the zeta-function, paving the way for the future proof of the prime number theorem, and also making it clear why the location of the "zeta zeros" is so crucial. Finally, we meet the Riemann hypothesis itself which is the most desirable, and also the simplest, explanation for the location of these zeros.

After some time on the ideas of Stieltjes for proving the Riemann hypothesis by showing that the Möbius function behaves like a random walk, we reach Hadamard and de la Vallée Poussin and the proof of the prime number theorem. We then turn the century. As it must, the century begins with Hilbert's problems. But then we come to the eighth problem, and back to the Riemann hypothesis.

The first seventy years of the century saw the work of H. Bohr and E. Landau, of Hardy and Littlewood, of Cramér, of Siegel and Riemann's *Nachlass*, of Weil and "a Riemann hypothesis that is true" and of Selberg, who would later be described as "the éminence grise of the Riemann hypothesis". This summary happens in about twenty-five pages. Are my hackles showing? Not really. There are many deep and beautiful theorems from this period but perhaps not so much possibility to explain more than a few of the most striking statements to this audience.

The past thirty-five years get substantially more press. And why not? There is lots of sexy stuff happening. We start with the chance (Chowla-engineered) meeting of Dyson and Montgomery and the pair-correlation of the zeros, which ignited the revival of the Hilbert-Pólya dream of a spectral interpretation of the zeros and the consequent resolution of the Riemann hypothesis. With Odlyzko's impressive computational evidence and further theoretical evidence provided by the higher correlations (Rudnick and Sarnak) and the function field analogues (Katz and Sarnak), it appears more and more as though the zeros are behaving like the eigenvalues of random matrices from certain families studied by the physicists! Does this help us to prove the Riemann hypothesis any more than Stieltjes was helped by the fact that the Möbius function appears to behave like the toss of a random coin? Well, at least this connection is not so straight-forward. We have “quantum chaos”, we have “life at the semi-classical limit”, we have “billiards in the Poincaré disk”.

The reviewer is not the only one to be somewhat skeptical. “Sarnak is among the first to say that in spite of all the exciting and beautiful mathematics and physics coming out of recent investigations into quantum chaos it is ‘naive’ to think that this work will culminate in the discovery of a physical system whose energies produce the zeta zeros, and thus a proof of the Riemann hypothesis.”

Still we go on, to noncommutative geometry, the central limit theorem, the Painlevé differential equations, and the shuffling of decks of cards! Why are we doing this?

“In this web of connections we truly see the stature of the Riemann hypothesis. A great problem of mathematics becomes an intellectual nexus, providing a bridge across subjects and connecting seemingly disparate ideas…And finally, with its relevance to almost all of mathematics laid bare, almost every mathematician can have a chance to dream of contributing to, and (dare we say!) even settling, this most important open problem in mathematics that is the Riemann hypothesis.”

Inside this quote is one wisp: “a chance to dream”. Now, that is something to which I can relate!

And, on the seventh day God rested. But, by afternoon she had become a little bored and decided to play a game. She set out to create the story of the primes. However, even being God, she did something that you or I would do. She used the tools with which she was familiar, the same tools she had been using all week long to create the universe.