The Pea and the Sun: A Mathematical Paradox

Reviewed by Péter Komjáth

The Pea and the Sun: A Mathematical Paradox
Leonard M. Wapner
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It is a welcome fact that recent popular books on mathematics cover not just elementary topics, but some of the more advanced ones, such as the Riemann hypothesis, the Hilbert problems, and the million-dollar problems.

Leonard Wapner chose the Banach-Tarski paradox (BTP) to write about. An interesting choice. It is one of the (very) few great mathematical results that can be fully understood by anyone: a solid ball can be split into a finite number of parts that can be rearranged into a similar decomposition of *two* balls, both as large as the original. A similar proof shows that any two bodies in 3-space are likewise equidecomposable, assuming they are bounded and both have nonempty interior (this gives the title of the book). This result, which appeared in the 1924 volume of *Fundamenta Mathematicæ*, was preceded in 1914 by the following result of Hausdorff. There is a cocountable subset of S^2 that can be decomposed as $A \cup B \cup C$ such that A, B, C, and $A \cup B$ are all congruent. As it already has the character of saying 1 = 2 and as the proof of BTP leans heavily on its proof, the latter could perhaps be called the Hausdorff-Banach-Tarski paradox.

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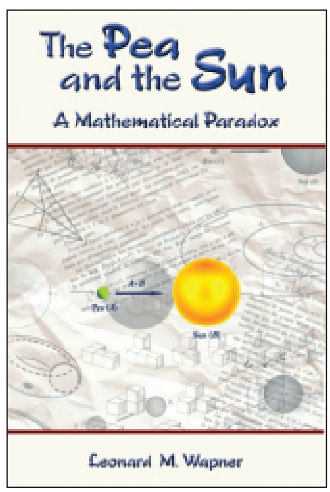

Although the BTP is known to virtually everyone who works in mathematics as a teacher or researcher, its proof is not. It is not a very hard proof, yet it is not easy to teach, say, to undergraduates (I just did that a few months ago). The problem is that it contains several technical elements that are standard in advanced mathematics but are strange at first sight, unfriendly to the beginner, and it takes time to get used to them: the axiom of choice, free groups, orbits, and the statement that there are two independent rotations around the origin in 3-space.

The standard reference on this topic is Wagon's monograph [2]. A good introduction for undergraduates can be found in Laczkovich's book [1].

The decision to write a popular book on the BTP allows several possibilities. One could write an entertaining book stating the theorem, describing its history, with easy, witty remarks. Or one could write a book that contains an elementary introduction to the corresponding mathematical notions. Or, finally, one could write a book with the full proof in an available form. Wapner admirably carried out his task by creating a book that has all three features.

The first chapter describes the lives of Cantor, Banach, Tarski, Gödel, and Cohen, and, in some detail, the work of Cantor and Gödel.

Chapter 2 contains paradoxes of various kinds: logical, physical, geometrical, and others.

The third chapter collects the necessary preliminaries. Introductions to set theory, spatial congruences, and matrix arithmetic are followed by some easy examples of equidecomposability.

Chapter 4 continues with more mature mathematics. After some remarks on infinity and fractional dimension, two key elements of the proof are given: how to eat up a countable subset of the sphere (that is, S^2 is equidecomposable into each of its cocountable subsets) and the Vitali paradox. Mazurkiewicz's countable (nonempty) planar set is presented; it has a 2-part decomposition in which both parts are congruent to the full set.

Chapter 5 contains the proofs of the BTP and the Schröder-Bernstein theorem (in the form that if two sets are mutually equidecomposable into a subset of the other, then they are equidecomposable).

The sixth chapter analyzes the ways of handling a paradox (declaring it fallacious, accepting it, or reinterpreting it).

Chapter 7 contains some interesting speculations about the BTP holding in the real universe.

The last chapter surveys some recent results and problems of mathematics.

The book is well written and entertaining. It teaches a lot of things about science, mathematics, infinity. The various steps of the proof of BTP along with several other theorems are carefully and clearly presented. I like the way Wapner treats free groups (actually finitely presented groups, for the author uses, as the original paper does, the group $\langle x, y | x^2 = y^3 = 1 \rangle$).

In the first chapter the author chose to describe the lives of Cantor, Banach, Tarski, Gödel, and Cohen. Cantor appears here as the creator of set theory, Banach and Tarski as inventors of the BTP, Gödel and Cohen as the ones who cleared up the consistency and independence of the axiom of choice, the most important element of the BTP. Indeed, this is a very good introduction to set theory, but Gödel and Cohen are not that closely connected to the main topic of the book. Further, the biographies of Cantor and Gödel are easily available in many places, so it would have been better to describe the lives of some lesser-known people instead.

The book mentions several interesting variants of the BTP, for example the Dougherty-Foreman result claiming that given two nonempty, bounded open sets in 3-space, one can find respective dense subsets that can be equidecomposed via open sets. They applied this to prove a conjecture of Marczewski, claiming that the BTP can be executed via pieces with the property of Baire (of course, measurability is out of the question). Another one is the Tarski Circle-Squaring Problem, now Laczkovich's theorem, that a square and a disc in the plane are equidecomposable, assuming they have the same area. What I miss is the mention of two other theorems which, admittedly, speak to the existence of sets and not to equidecomposability, yet the character of the results and the proofs put them into the same topic. One is Mycielski's set A in 3-space that has the property that if B is any set such that the symmetric difference of A and B is countable, then B is congruent to A. The other is Steve Jackson and Dan Mauldin's very technical and very clever recent solution of the Steinhaus problem: there is a planar set C with the property that every congruent copy of C contains exactly one lattice point (i.e., a point with integer coordinates).

What I also miss is the point that the property of $\mathbf{R}^3$ that makes duplication possible is the complexity of the congruence group: it contains free groups, while the congruences of $\mathbf{R}^2$ form a solvable group. There is, therefore, a finitely additive and congruence-invariant extension of the Lebesgue measure to all planar sets, and the nonexistence of a similar extension in space is an easy corollary of the BTP. The fact that under very general conditions there is duplication exactly when there is no general invariant measure is beautifully presented in Wagon's book.

Some readers may not realize that the hilarious quotation from St. Augustine on p. 1 is actually against astrologers ("The good Christian should beware of mathematicians, and all those who make empty prophecies. The danger already exists that mathematicians have made a covenant with the devil to darken the spirit and to confine man in the bonds of Hell").

I found a few typos: p. 77, l.7, $a \in f(A)$ should be $a \in f(a)$; p. 85, in the formula, the second = should be a $\cdot$; p. 126, l.8, $c - \frac{1}{2} = c$ should be $c \cdot \frac{1}{2} = c$.

References

[1] M. Laczkovich, *Conjecture and Proof*, Classroom Resource Materials Series, MAA, 2001.

[2] S. Wagon, *The Banach-Tarski Paradox*, Cambridge University Press, 1985.