WHAT IS...

a Bad End?

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A complex-valued function F defined in an open subset of the complex plane is holomorphic if it satisfies the Cauchy-Riemann equation: $\partial_{\bar{z}}F = \frac{1}{2}(\partial_x + i\partial_y)F = 0$. Similarly, a C^1-function defined in an open subset U of $\mathbb{C}^n$ is holomorphic if its restriction to every complex line passing through U is holomorphic. The behavior of the boundary values of holomorphic functions defined in domains in $\mathbb{C}^n$, or more generally on complex manifolds with boundary, is a problem of enduring interest in complex analysis. When viewed from a slightly different perspective, questions about boundary values of holomorphic functions lead to geometric questions about the boundaries of complex manifolds themselves.

If f is a continuous function defined on the boundary of the unit disk D then it is easy to tell whether or not there is a holomorphic function F defined in $\overline{D}$ with $F \restriction_{bD} = f$. We let $\Pi_+ f$ denote the projection of f onto the Fourier components with non-negative frequencies; f is the restriction to the boundary of a holomorphic function defined in D if and only if $\Pi_+ f = f$. We let H_+ denote the image of Π_+. Using the Riemann mapping theorem, we can give a similar answer for any compact, simply connected domain $\Omega \subset \mathbb{C}$ with a smooth boundary γ. Let $\varphi : D \to \Omega$ be a Riemann map. A function f defined on γ is the boundary value of a holomorphic function defined in Ω if and only if $\varphi^* f = f \circ \varphi \in H_+$.

Cauchy's theorem tells us that, if f is the boundary value of a holomorphic function defined on a smoothly bounded domain $\Omega \subset \mathbb{C}$, then $\int_{b\Omega} f(z)dz = 0$. This condition allows us to examine the following geometric question: Let γ be a smooth oriented curve in $\mathbb{C}^n$. Does there exist a 1-dimensional analytic variety $X \hookrightarrow \mathbb{C}^n$ with $bX = \gamma$? Suppose that such a variety exists, and let $(p_1, \dots, p_n)$ be an n-tuple of polynomials in $(z_1, \dots, z_n)$. Using Cauchy's theorem (via Stokes' formula) we conclude that, if $\omega = \sum_{j=1}^n p_j(z)dz_j$, then $\int_\gamma \omega = \int_X \overline{\partial}\omega = 0$. The left hand side is a computation done entirely on γ, and so the vanishing of these integrals provides necessary conditions for the existence of the variety X. Starting with work of Wermer, Bishop, and Alexander, and continued by Harvey and Lawson, it was shown that these are also sufficient conditions [2].

Colloquially the boundary components of a manifold with boundary are its *ends* (though this is somewhat at variance with the usual usage in topology). From the perspective of complex geometry, a curve $\gamma \subset \mathbb{C}^n$ is a *good end* if there is an analytic variety $X \subset \mathbb{C}^n$ with boundary equal to γ; otherwise it is a *bad end*. It is clear that in order for a curve to be a good end it must satisfy infinitely many independent conditions. Hence most curves are bad ends, and the property of being a good end is unstable under small deformations of the embedding.

Another way to phrase these results is to let H_+^γ be the closure, in the L^2-norm, of the restrictions of polynomials to γ. We let $h : bD \to \gamma$ be an orientation-preserving diffeomorphism and $H_+^{\gamma,h} = h^*(H_+^\gamma)$. The curve $\gamma \hookrightarrow \mathbb{C}^n$ is a good end if and only if the map $\Pi_+ : H_+^{\gamma,h} \to H_+$ has a finite-dimensional kernel and co-kernel; in other words, if this restriction is a Fredholm map. The Fredholm index, when it is finite, is related to the genus and singularities of X.

Now let Ω be a bounded domain in $\mathbb{C}^2$ with a smooth boundary $b\Omega$, and suppose that F is a holomorphic function defined in a neighborhood of Ω. This implies that $\partial_{\bar{z}}F = \partial_{\bar{w}}F = 0$ in $\overline{\Omega}$. If ρ is a function that is negative inside of Ω and vanishes simply on $b\Omega$, then the complex vector field $\overline{Z}_{b\Omega} = \rho_{\bar{w}}\partial_{\bar{z}} - \rho_{\bar{z}}\partial_{\bar{w}}$ has its real and imaginary parts tangent to $b\Omega$. If F is holomorphic in Ω and smooth up the the boundary, then $\overline{Z}_{b\Omega}(F \restriction_{b\Omega}) = 0$. Even if f is defined only on $b\Omega$, we can compute $\overline{Z}_{b\Omega} f$. For f to be the boundary

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value of a holomorphic function in Ω, it is necessary and sufficient that $\overline{Z}_{b\Omega}f = 0$. A function satisfying this equation is called a *CR-function*.

The complex structure on $\mathbb{C}^2$ induces a geometric structure on a hypersurface $Y \hookrightarrow \mathbb{C}^2$: The complex vector bundle $T^{0,1}\mathbb{C}^2$ is spanned at each point by $\partial_{\bar{z}}, \partial_{\bar{w}}$. If Y is a hypersurface, then for each $y \in Y$ there is a complex line $T_y^{0,1}Y \subset T_y^{0,1}\mathbb{C}^2$, consisting of vectors whose real and imaginary parts are tangent to Y. These lines fit together smoothly to define a complex vector bundle $T^{0,1}Y \to Y$. This is called a *CR-structure* on Y. For X, a complex 2-dimensional manifold with boundary, the complex structure on X defines, in the same way, a CR-structure on bX.

If $Y \hookrightarrow \mathbb{C}^n$ is a real 3-dimensional submanifold of $\mathbb{C}^n$, then for each $y \in Y$ we let $T_y^{0,1}Y = T_y^{0,1}\mathbb{C}^n \cap TY_y \otimes \mathbb{C}$. In order for there to be a complex surface $X \hookrightarrow \mathbb{C}^n$ with boundary equal to Y it is clearly necessary that $\dim T_y^{0,1}Y = 1$, for every $y \in Y$. Such a submanifold is called *maximally complex*. Harvey and Lawson showed that this is also a sufficient condition [2]. By analogy with the 1-dimensional case, we say that a 3-manifold embedded in $\mathbb{C}^n$ is a good end provided it is the boundary of a complex surface. If $n > 2$, then maximal complexity is not generic: for most y, $T_y^{0,1}Y$ is just the zero vector. If $Y \hookrightarrow \mathbb{C}^n$ is a good end, then, as before, wiggling the embedding a little usually produces a bad end.

A CR-structure can be defined intrinsically on a 3-manifold: If Y is a 3-manifold and $H \subset TY$ is a plane field, then a smoothly varying choice of complex structure on the fibers of H defines a CR-structure. A CR-structure is a splitting of $H \otimes \mathbb{C}$ into two conjugate sub-bundles $H \otimes \mathbb{C} = T^{0,1}Y \oplus T^{1,0}Y$. We define a differential operator on Y, analogous to the $\overline{\partial}$-operator: $\bar{\partial}_b f = df \restriction_{T^{0,1}Y}$. A function f is a CR-function if $\bar{\partial}_b f = 0$. Let θ be a non-vanishing one-form defined on Y so that $H = \ker\theta$. The plane field defines a contact structure if $\theta \wedge d\theta$ is non-vanishing. For a domain this is the same as requiring that the Hermitian form, $\mathcal{L}(Z, W) = \partial\overline{\partial}\rho(Z, \overline{W})$ for $Z, W \in T^{1,0}b\Omega$, be positive definite. This form, called the *Levi form*, can be defined intrinsically by setting $\mathcal{L}(Z, W) = id\theta(Z, \overline{W})$. The CR-manifold $(Y, T^{0,1}Y)$ is called *strictly pseudoconvex* if $\mathcal{L}$ is positive definite.

Suppose that $(Y, T^{0,1}Y)$ is a compact strictly pseudoconvex CR-manifold. If there exists a compact, complex manifold X with strictly pseudoconvex boundary $(Y, T^{0,1}Y)$ then we say that Y is a *fillable* CR-manifold. A fillable CR-manifold is a *good end*, a nonfillable CR-manifold is a *bad end*. On a fillable CR-manifold, the CR-functions separate points.

It turns out that "most" strictly pseudoconvex 3-manifolds are bad ends, see [1]. It is easy to write down an explicit example of a bad end: The CR-structure on the unit sphere $S^3 \subset \mathbb{C}^2$ is generated by $\overline{Z} = w\partial_{\bar{z}} - z\partial_{\bar{w}}$. If $\epsilon \in \mathbb{C}$ with $0 < |\epsilon| < 1$, then $\overline{Z}_\epsilon = \overline{Z} + \epsilon Z$ defines a strictly pseudoconvex CR-structure on S^3 that does not arise as the strictly pseudoconvex boundary of any complex manifold. This seminal example appeared in the early 1960s in the work of A. Andreotti, H. Grauert, and H. Rossi. Dan Burns showed that all global solutions to $\overline{Z}_\epsilon f = 0$ are even functions, and therefore the CR-functions defined by $\overline{Z}_\epsilon$ do not separate points.

Suppose that $(Y, T^{0,1}Y)$ is a good end. The CR-structures on (Y, H) can be described by "Beltrami-differentials," that is, sections μ of the bundle $\mathrm{Hom}(T^{0,1}Y, T^{1,0}Y)$, with $\|\mu\|_\infty < 1$. We would like to know: What is the set of μ such that $(Y, {}^{\mu}T^{0,1}Y)$ is a good end? This problem is difficult because the set of deformations is infinite-dimensional, and there are infinitely many conditions that must be satisfied for a deformation of a good end to be a good end. Nonetheless, in some cases, this question now has a good answer.

Let (Y, H) be a contact 3-manifold and $(Y, T^{0,1}Y)$ a *fillable*, strictly pseudoconvex CR-structure on (Y, H). For μ a deformation of the CR-structure, we denote the corresponding $\bar{\partial}_b$-operator by $\bar{\partial}_b^\mu$. Let S^μ denote the orthogonal projection onto the $\ker \bar{\partial}_b^\mu$. Using results of J. J. Kohn and L. Boutet de Monvel, one can show that μ defines a good end if and only if the restriction $S^\mu : \ker \bar{\partial}_b^0 \longrightarrow \ker \bar{\partial}_b^\mu$ is a Fredholm operator [1]. We let $\text{R-Ind}(S^0, S^\mu)$ denote the index of this operator. Using a beautiful compactification trick, L. Lempert showed that if Ω is a strictly pseudoconvex domain in $\mathbb{C}^2$, then any fillable, small deformation of the CR-structure on $b\Omega$ can be obtained by wiggling the embedding of $b\Omega$ in $\mathbb{C}^2$. Hence, the entire algebra of CR-functions on $b\Omega$ is stable under small embeddable deformations, and so in this case $\text{R-Ind}(S^0, S^\mu)$ equals 0 or ∞. In [1] it is conjectured that, for any fillable strictly pseudoconvex 3-dimensional CR-manifolds, among small fillable deformations, $\text{R-Ind}(S^0, S^\mu)$ assumes only finitely many values. Among other things, this would imply that the set of embeddable deformations is a closed subset. This conjecture has been established in many cases.

Since the work of Kohn and Rossi, analysis on CR-manifolds has been a major theme in several complex variables. M. Kuranishi used the CR-manifolds, defined as links of isolated analytic singularities, as the starting point for his construction of versal deformations. With the recent advances in the contact (symplectic) geometry of 3-(4-)manifolds, initiated by the pioneering work of Y. Eliashberg and M. Gromov, there has been a resurgence of interest in the problem of filling 3-dimensional CR-manifolds, see [3].

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