

2006 Gauss Prize



Kiyoshi Itô

The Work of Kiyoshi Itô

The following announcement of the Gauss Prize is adapted from an IMU news release.

Carl Friedrich Gauss (1777–1855), known as “princeps mathematicorum” (“prince of mathematicians”), united within his person two sides of mathematics. He not only achieved great progress in number theory, called the “queen of mathematics” because it was glorious as well as far from any real-world applications, a statement that was valid until a few decades ago. He also created what is today called the “least-squares fit”, a method that is applied every time you have to deal with real-world problems, in particular measuring inaccuracies.

The newly created Gauss Prize honors “mathematical research that has had an impact outside mathematics—either in technology, in business, or simply in people’s everyday lives”. The prize is awarded jointly by the Deutsche Mathematiker-Vereinigung (DMV, German Mathematical Society) and the International Mathematical Union (IMU) and administered by the DMV. It consists of a medal and a monetary award (currently valued at 10,000 €, approximately US\$13,000). The source of the prize is the surplus from the ICM held in Berlin in 1998. The prize was awarded for the first time at this year’s ICM in Madrid.

Kiyoshi Itô’s theoretical work has had an enormous impact and shows that the route from real-world phenomena, to abstract mathematical description, and back to real-world application, is often long and circuitous.

In Itô’s case, the route begins with a look in a microscope at water containing pollen grains or

dust particles that move around in an erratic way. Today, this strange dance is called Brownian motion after the Scotch botanist Robert Brown, who observed it in 1827 and gave a detailed description. The motion of the particles is the net effect of a large number of collisions between the particles and the water molecules. It was Albert Einstein (1879–1955) who formulated a mathematical model of Brownian motion in one of his three papers of 1905, each of which amounted to a revolution in science. Norbert Wiener (1894–1964), better known as the founder of cybernetics, followed in 1923 with a proof that Einstein’s model was mathematically sound.

It turned out that Einstein and Wiener had found a mathematical idealization of pure chance. A Wiener process can be used to describe how dust particles move in water, how lines at a checkout counter grow or shrink, or how the price of a share of stock fluctuates. This model is about as universal as Gauss’s normal distribution, which turns up every time a quantity is influenced by many independent perturbations.

This idealization, however, came at a price. Physicists normally assume that nature behaves smoothly in some sense, but a Wiener process violates this assumption in a fundamental way. The particle’s path is infinitely wiggly, or nowhere differentiable in technical terms. Moreover, it is infinitely long. Just a hundred years ago, mathematicians used to turn away in horror from those “monsters”, as they are intractable with the classical tools to handle curves.

One of those classical tools is the integral. Beginning in 1942, Itô reformulated the classical way of defining the integral and created a new concept called the “stochastic integral”. This work included calculation rules and a solution theory for stochastic differential equations. An ordinary differential equation is the form mathematicians use to describe the motion of a particle under a known (deterministic) force; a stochastic differential equation incorporates forces that depend on, for instance, a Wiener process.

A stochastic integral in itself can never be the final solution of a problem. You cannot know a

random process in advance—if you could, it wouldn't be random—so it's useless to ask where a point subject to a stochastic differential equation will be five minutes from now, or when it will cross a given line for the first time. However, Itô's method provides probabilities of the occurrence of events of that kind.

Consider a gambler's assets in a game like roulette, or the value of an investor's portfolio. The gambler and the investor have a vital interest in knowing when a certain value, which depends on chance as well as on their own actions, first crosses the zero line. Risk-averse players, in particular bankers, want to direct their own actions so as to minimize the effects of chance.

This is the idea underlying financial instruments like options and futures. A call option amounts to a bet on a future event, such as the price of a share reaching a certain value. The bank offering the option must make provisions to have the money available on the day it might lose the bet; the cost of those provisions is the price of the option that the bank charges the customer. The bank, however, is free to take these provisions, depending on the actual share price, at any time from now on up to the time the option is due. The option price can be expressed and calculated by a stochastic integral.

Today the most influential applications of Itô's theory are found in finance. At the beginning of the 1970s, the economists Fischer Black, Robert C. Merton, and Myron S. Scholes found an explicit formula to calculate the price of an option. Today the Black-Scholes formula, which contains only known data, underlies almost all financial transactions that involve options or futures; moreover it won Merton and Scholes the Nobel Prize in Economics in 1997 (Black died in 1995).

But Itô's theory is sufficiently abstract to serve completely different needs. It applies also to the size of a population of living organisms, to the frequency of a certain allele within the gene pool of a population, or to even more complex biological quantities. Chance is not completely blind in these cases: The average fluctuation of a population size is not a constant but proportional to the size, and the frequency of two alleles that occupy about half of the population tends to change more rapidly than if one of them is close to extinction. For such applications the concept of a Wiener process had to be generalized, a task that would have been almost impossible without Itô's theoretical framework. Biologists can now assess the probability with which a gene will dominate the whole population or a species will survive.

These generalizations, in turn, came in handy for the economists. In 1985 John C. Cox, Stephen A. Ross, and Jonathan E. Ingersoll found a mathematical model for the time evolution of interest rates that has now become standard. Stephen L.

Heston generalized the Black-Scholes model in 1993 so as to bring it closer to reality.

On the other hand, it took mathematicians themselves quite a while to appreciate the importance of Itô's results. This is partly due to Japan's isolation during World War II. Itô lectured on his achievements during a stay at the Institute for Advanced Study in Princeton in 1954–1956.

Moreover, there was a competing theory available to describe the effects of pure chance on a more global level. If you want to know how a drop of ink disperses in water, you can either try to follow the Brownian motion of single ink particles, or you can consider both ink and water as a continuum and formulate their motion in terms of a partial differential equation—a diffusion equation in this case. As both methods describe the same physical phenomenon, they should lead to the same results, so there should be some connection between them. It took some time to bring this connection to light, but it has already been put to good use. The Black-Scholes formula contains the solution of a diffusion equation.

Today, there is no doubt that stochastic analysis is a rich, important, and fruitful branch of mathematics with a formidable impact on technology, business, and everyday life.

Biographical Sketch

Kiyoshi Itô was born on September 7, 1915, in Japan. He graduated from the Imperial University of Tokyo in 1938. He served in the Statistics Bureau of the Cabinet Secretariat of Japan from 1939 until 1943, when he became an assistant professor in the Faculty of Science at Nagoya Imperial University. In 1945 he received his D.Sc. from the Imperial University of Tokyo. He became a professor at Kyoto University in 1952 and professor emeritus in 1979. He then joined the faculty of Gakushin University and retired from there in 1985. He held visiting positions at the Institute for Advanced Study (1954–56), Aarhus University (1966–69), and Cornell University (1969–75). He served as director of the Research Institute for Mathematical Sciences at Kyoto University from 1976 to 1979.

Itô has received some of Japan's highest honors, including the Asahi Prize (1978), the Imperial Prize and the Japan Academy Prize (1978), and the Fujiwara Prize (1987). He also received the Wolf Prize in 1987 and the Kyoto Prize in 1998. He is an associate foreign member of the Académie des Sciences of France, a member of the Japan Academy, and a foreign member of the U.S. National Academy of Sciences. He has received honorary degrees from the Eidgenössisches Technische Hochschule in Zurich and the University of Warwick.

—Allyn Jackson