# Turing and the Riemann Hypothesis

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Alan Turing's final research paper[1] [11] described a numerical method for verifying the Riemann hypothesis and its implementation on the Manchester Mark I, one of the earliest general purpose digital computers. Turing writes in his introduction

> The calculations had been planned some time in advance, but had in fact to be carried out in great haste. If it had not been for the fact that the computer remained in serviceable condition for an unusually long period from 3 p.m. one afternoon to 8 a.m. the following morning it is probable that the calculations would never have been done at all. As it was, the interval $2\pi.63^2 < t < 2\pi.64^2$ was investigated during that period, and very little more was accomplished.

The modesty of this last sentence notwithstanding, Turing's paper is an important contribution to number theory that continues to have relevance today; indeed, we are fortunate that the Manchester computer remained serviceable for so long on that day, for otherwise Turing may never have published his results! The goal of this article is to describe the method and some recent developments in a historical context.

## Background

We begin with a very brief introduction to the Riemann hypothesis and some associated computational aspects; for a full account, including its importance in number theory and recent attempts at proof, see the excellent survey article by Conrey [4].

The $\zeta$-function is defined for complex numbers $s$ with real part $\Re(s) > 1$ by the series

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{1}$$

which converges absolutely. As discovered by Riemann, it has analytic continuation to $\mathbb{C}$, except for a simple pole at $s = 1$. Moreover, a functional equation relates the values at $s$ and $1 - s$: If $\gamma(s) := \pi^{-s/2}\Gamma(s/2)$ and $\Lambda(s) := \gamma(s)\zeta(s)$ then

$$\Lambda(s) = \Lambda(1 - s). \tag{2}$$

The Riemann hypothesis is the conjecture that all zeros of the modified function $\Lambda(s)$ have real part exactly $\frac{1}{2}$. All that is known at present, however, is that the real parts lie in the open interval $(0, 1)$.

Since $\Lambda(s)$ is real for $s$ on the real axis, the zeros come in complex-conjugate pairs $s, \bar{s}$, so it suffices to consider here only the ones in the upper half plane. The number of zeros with imaginary part $\Im(s) \in (0, t]$, denoted by $N(t)$, is roughly $\theta(t)/\pi + 1$, where $\theta(t)$ is the phase of $\gamma(\frac{1}{2} + it)$, i.e., the continuous function such that $\theta(0) = 0$ and

$$\gamma(\tfrac{1}{2} + it) = |\gamma(\tfrac{1}{2} + it)|e^{i\theta(t)}. \tag{3}$$

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[1] *A popular account of some of his ideas on computability appeared the following year in [12].*

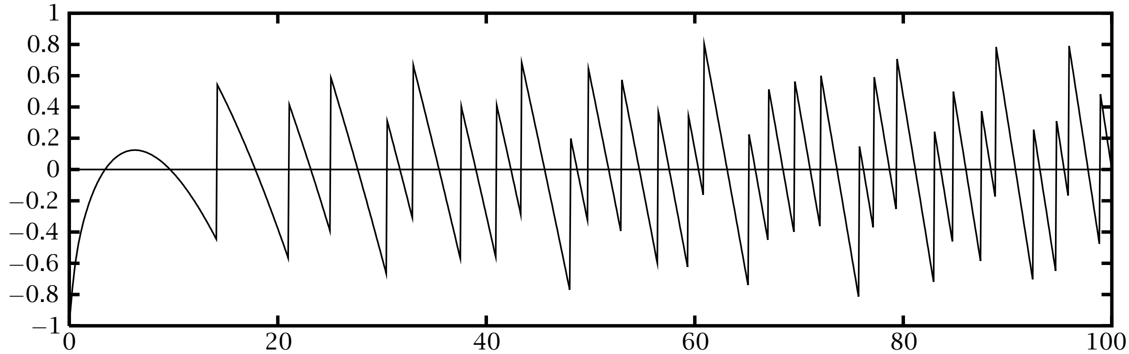


**Figure 1.** $S(t)$.

This may be computed quickly for large $t > 0$ by the asymptotic formula

$$\frac{\theta(t)}{\pi} + 1 \approx \frac{t}{2\pi} \log \frac{t}{2\pi e} + \frac{7}{8}. \tag{4}$$

In particular, $\Lambda(s)$ has many zeros.

The difference

$$S(t) := N(t) - \left(\frac{\theta(t)}{\pi} + 1\right) \tag{5}$$

between $N(t)$ and the expected count is a function that seems to vary unpredictably, as can be seen in Figure 1. This strange[2] behavior, and our incomplete understanding of it, lies at the heart of what makes the Riemann hypothesis a difficult problem and numerical computation a useful tool.

An important ingredient when doing numerics is an algorithm for computing the $\Lambda$-function at arguments $s = \frac{1}{2} + it$. However, since $|\gamma(\frac{1}{2} + it)|$ decreases exponentially for large $t$, one usually works instead with the function $Z(t) := e^{i\theta(t)}\zeta(\frac{1}{2} + it)$, which is real-valued for $t \in \mathbb{R}$ and has the same zeros as $\Lambda(\frac{1}{2} + it)$. A formula for $Z(t)$, known to Riemann and rediscovered by Siegel, is the following.

$$Z(t) \approx 2 \sum_{n=1}^{\lfloor \sqrt{t/2\pi} \rfloor} n^{-1/2} \cos(\theta(t) - t \log n). \tag{6}$$

The error of the approximation is no worse than $O(t^{-1/4})$, so that (6) becomes more accurate for larger $t$; moreover, there is an asymptotic expansion for the error term, giving better accuracy yet. For small values of $t$, the error in the Riemann-Siegel formula is too large, and one usually prefers a different technique, known as Euler-Maclaurin summation, which allows for high accuracy at the expense of a longer running time.

## Turing's Interest in the Riemann Hypothesis

According to Hodges' definitive biography [6], Turing became interested in the Riemann hypothesis while still a student. Curiously, he seems to have believed it to be false; indeed, it is clear from [11] that he had hoped to find a counterexample. In 1939, back in Cambridge, he conceived of an analog machine to aid with the calculations necessary for numerically checking the hypothesis. The design of the machine, whose blueprint is reproduced on the cover and pages 1186–1187, called for an assembly of eighty gears of precise ratios and a counterweight, which would physically perform the sum in (6). Turing won a grant for £40 from the Royal Society to cover the cost of its construction and got as far as manually cutting some of the gears, which would often end up on the floor of his room. However, World War II intervened before the work was completed, and Turing would have other important contributions to make.

By the time that he returned to the problem, in June 1950, digital computers had advanced to the point that it was practical, if only barely so, to consider much more than was possible with any analog machine—testing the Riemann hypothesis algorithmically, with no human intervention. Indeed, this is an important aspect of Turing's method which should not be overlooked.[3] Although Turing's numerical results were modest—Titchmarsh had by 1936 achieved nearly the same range by more conventional means—it wasn't long before Lehmer extended his calculations to ranges well out of reach of hand computation. However, that this would be the case may have been far from obvious in 1950; few at the time could have anticipated the economies of scale in speed, reliability, and availability of computing technology that would be achieved, forever rendering human computers obsolete.[4] As it was, the practical issues that Turing faced, described in detail in [11], were formidable compared to today's technology.

[2] *One might substitute the word* random *here: Selberg showed that the values of $S(t)/\sqrt{\log\log t}$, as $t \to \infty$, are normally distributed.*

[3] *It was also part of the larger consideration of the extent to which machines could think and act autonomously, a question that captured Turing's keen interest.*

[4] *Up to the 1940s, the word* computer *referred to a human who performed computations with the aid of a calculating device. With the advent of electronic machines and stored programs, the job of the human shifted to that of* programmer. *Turing employs both the original and modern usages in [11].*

Hodges speculates that Turing rushed [11] to publication, worried that he would be sent to prison. What is clear is that he was dissatisfied with the results. Unfortunately, we will never know the true extent of his intentions.

## The Method

Turing's approach follows earlier computations by Gram, Backlund, Hutchinson, and Titchmarsh. First, one locates many zeros on the line $\Re(s) = \frac{1}{2}$ up to a given height $T$ by computing $Z(t)$ and noting its changes of sign. Second, one shows that the computed list of zeros is complete (meaning that the Riemann hypothesis is true up to height $T$), by determining $N(T)$ via an auxiliary computation.

Turing made contributions to both aspects. In [10] he introduced an algorithm for computing the $Z$-function that was intended to be used in the intermediate range, between those of the Riemann-Siegel formula and Euler-Maclaurin summation. However, with better error terms known today and improvements in computing technology, that gap has been closed otherwise. On the other hand, his technique for determining $N(T)$ was of more lasting value and is what is usually meant when referring simply to "Turing's method".

The authors prior to Turing used an ad hoc approach, described in detail by Edwards [5, §6.7]; it was both computationally expensive and not guaranteed to work for any given $T$. Turing's method relies instead on the fact, first due to Littlewood, that the average value of $S(t)$, for $t$ ranging over the interval $[0, T]$, tends to 0 as $T$ grows. Thus, the graph of $S(t)$ tends to oscillate around 0, as is visible in Figure 1. Now, if one imagines plotting Figure 1 using equation (5) and the *measured* data for $N(t)$, any zeros that had been missed would skew the graph, i.e., it would begin to oscillate around an integer less than 0, corresponding to the number of missing zeros. (Note that when locating zeros by sign changes, one always misses an even number of them.)

To make this precise, one needs an explicit form of Littlewood's theorem. This is one of the main results of [11], where Turing proved the estimate

$$\left| \int_T^{T+h} S(t)\,dt \right| \le 2.3 + 0.128 \log \frac{T+h}{2\pi}, \tag{7}$$

valid for all $h > 0$ and $T > 168\pi$. With (7) in hand, one entertains the hypothesis that at least one zero up to height $T$ has been overlooked and computes the integral using the numerically measured values of $N(t)$, with the extra zero thrown in. If it turns out that there really is no missing zero, then (7) will be contradicted with a value of $h$ on the order of $c \log T$ for a small number $c$. Thus, roughly speaking, in order to certify complete the list of zeros up to $T$, one needs knowledge of the $\zeta$-function up to height about $T + c \log T$. When $T$ is large, that is a negligible price to pay compared to the total computation.

Turing's proof of (7) is elegant and remains essentially unchanged in all subsequent generalizations. The bound is not sharp,[5] however, and the constants were later improved by Lehman, who also corrected a few errors in the details. Nevertheless, (7) is more than sufficient for numerics; in fact, in modern verifications of the Riemann hypothesis, Turing's method is considered an automatic check, and one can concentrate on the business of locating the zeros as quickly as possible.

## Recent Developments

The more than half century following Turing's death has seen many developments in computational aspects of the Riemann hypothesis and related problems. In fact, Turing's method is arguably the first in a long line of papers in the area of computational analytic number theory; see [8] for a recent survey.

Concerning the Riemann hypothesis, an essentially optimal algorithm (in terms of speed) for computing the $\zeta$-function was developed by Odlyzko and Schönhage [7]. It uses the Fast Fourier Transform and computes many values of $Z(t)$ in *average* time $O(t^\varepsilon)$ per value, compared to the roughly $\sqrt{t}$ steps needed for a single evaluation using the Riemann-Siegel formula. The algorithm has led to computations of the $\zeta$-function on a much larger scale than Turing could have envisioned; in particular, the Riemann hypothesis has now been verified up to the ten trillionth zero! Turing's method remains a small but essential ingredient in those investigations.

Perhaps more importantly, the same computations have aided in the discovery of links between the $\zeta$-function and random matrix theory, which has in turn led to a flurry of recent work. A strong argument can be made that the eventual proof of the Riemann hypothesis will require a deeper understanding of this connection. See [4] for a description of these exciting developments.

In the same vein, number theorists today recognize that the $\zeta$-function is just one of a large class of important generating functions, known as $L$-functions. Many of the conjectures for $\zeta$, including the Riemann hypothesis and connections with

[5] *The coefficient of* $\log \frac{T+h}{2\pi}$ *in Turing's estimate is closely related to knowledge about the growth rate of* $Z(t)$ *as* $t \to \infty$. *The Lindelöf hypothesis, which is the conjecture* $Z(t) = O(t^\varepsilon)$, *is equivalent to the integral being* $o(\log(T+h))$ *as* $T + h \to \infty$. *The Riemann hypothesis, which in turn implies the Lindelöf hypothesis, yields the stronger bound* $O\left(\frac{\log(T+h)}{\log\log(T+h)^2}\right)$. *Heuristic arguments based on random matrix theory suggest that the true maximum size of the integral is closer to* $\sqrt{\log(T+h)}$.

random matrix theory, are expected to hold true for these functions as well. Very recently, Turing's method has been extended to all $L$-functions in [2]. It is interesting to note that few of the known techniques in analytic number theory apply in such wide generality; the fact that Turing's method does demonstrates how fundamental it is.

Finally, the novelty of Turing's method is further underscored by the fact that it was rediscovered some forty years later in the seemingly unrelated context of computing the spectrum of the Laplace operator on hyperbolic manifolds. This has had several applications in number theory and high energy physics; see [9] for a nice survey and [1] for an interesting application to cosmology. Unfortunately, the papers in the subject generally use the method without proper attribution to Turing. It would be good to have the record set straight. To that end, a rigorous treatment of the simplest example, much in the style of Turing, will appear in [3].

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