Book Review

The Shoelace Book

Reviewed by Colin Adams

The Shoelace Book: A Mathematical Guide to the Best (and Worst) Ways to Lace Your Shoes
Burkard Polster
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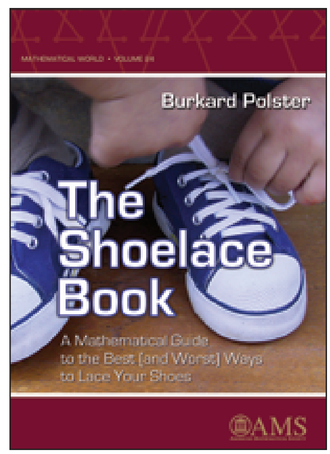
In 1999 two British physicists, Thomas Fink and Yong Mao, published an article in *Nature* about the various ways one might tie a necktie and the mathematics behind determining these options. This article received an immense amount of attention, with newspaper articles about it appearing around the globe. People were fascinated by this application of "science" to their everyday lives. A substantial portion of humanity gets up every morning and ties on a tie. It is a process that, once learned, becomes completely automatic, and is performed without the least thought.

But along come Fink and Mao, and suddenly, everyone is trying to look down at that knot just below their chin. Now people think about how they are tying that tie, and how they might tie it, and how one determines all the possibilities.

Subsequently, Fink and Mao published a slim book entitled *The 85 Ways to Tie a Tie*. At least for a while this was a hugely successful book, becoming the standard gift for every tie-toting office worker. Although the book does include a short appendix explaining the interpretation of necktie knots as random walks on a triangular lattice, in fact, the book is mostly about the history of knot tying and about the eighty-five ways that one can tie a tie, given the constraints that the authors have selected in terms of symmetry, aesthetics, and length of the tie. And although the appendix does a good job of making the mathematics sound sophisticated, in fact, it is completely elementary.



At the time this book came out, I remember thinking what a great idea to use neckties as a means to introduce knot theory. As a knot theorist, I kicked myself for not coming up with this obvious way to interest a general audience in certain aspects of mathematics.

In 2002 Burkard Polster published a short article in *Nature* about the mathematics of lacing one's shoes. And once again, stories about it appeared in major newspapers around the world. Once again, this was a subject that reached out to the widest of possible audiences. Other than very young children and a handful of Velcro lovers, who doesn't use shoelaces?

After hearing about Polster's article, the first thing you do is look down at your feet. There are your shoelaces, which you have not considered seriously for years. And you quickly find yourself thinking about the various ways you could be lacing them.

The Shoelace Book is Burkard Polster's expanded version of that article. In it, he discusses a wide variety of choices for lacing shoes, some with historical significance and many more that up to

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now, have not been made of use. We read about the crisscross lacing, which is easily the most common way to lace one's shoes. Then there is the zigzag lacing, which tradition had as the gentlemanly way to lace up oxfords. The more unusual lacings include the star, the serpent, the bowtie, the zigsag, the devil, and the angel lacings. Each of these is representative of certain classes of lacings, and here is where the mathematics comes in.

The goal is to model a real-world lacing that has been tied with some kind of knot at the top, the details of which we do not consider, only thinking of that knot as having turned the shoelace into a closed loop. To pose the questions carefully, we assume that a "mathematical shoe" has $2n$ eyelets, arranged in the plane in two vertical columns of n each, with adjacent eyelets in a given column separated by a vertical distance of h, called the *stretch* of the shoe, and horizontal pairs of eyelets in each row separated by a distance of 1. An *n-lacing* of the shoe consists of a closed path in the plane made up of $2n$ line segments whose endpoints are the $2n$ eyelets, with two line segments sharing each eyelet. The line segments that make up a lacing are split into the three classes of vertical, horizontal, and diagonal. The only restriction on the closed paths that we consider is that we do not allow both of the line segments sharing a given eyelet to be vertical, as then this eyelet would not contribute to pulling the two sides of the shoe together. The *length* of a lacing is the sum of the lengths of the segments that make it up.

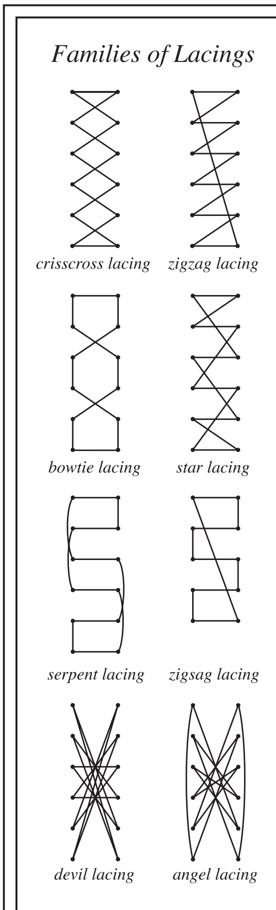
Lacings are then categorized into various classes. A lacing is *dense* if it does not contain any vertical segments. The crisscross, zigzag, and star lacings are all dense. A lacing is *straight* if it contains all possible horizontal segments. The zigzag, star, serpent, and zigsag lacings all fall into this category. A lacing is *superstraight* if it is straight and all nonhorizontal segments are vertical. Serpent lacings are superstraight. A lacing is *simple* if, when you start at a top eyelet and trace the lacing, you reach the bottom without any backtracking up and then, when you return to the top, you do so without any backtracking down. The crisscross, zigzag, star, bowtie, serpent, and zigsag lacings are all simple.

After this introductory material in Chapter One, the next chapter addresses one-column lacings. Given a straight lacing, we imagine pulling hard on the two ends of the lace before tying a knot in the top, and if our foot is narrow enough and our shoe flexible enough, the two columns of eyelets are pulled one on top of the other and the horizontal line segments become vertices. We now have a single column of n eyelets, and a closed path made up of n vertical segments which visits every eyelet once. This is a one-column lacing. The author determines the shortest and longest one-column lacings and the total number of one-column lacings.

The third chapter is devoted to deriving formulas for the number of lacings of the various types: general, dense, simple, straight, dense-and-straight, etc. Although some of the resultant formulas are simple enough and relatively straightforward, a few are not. Particularly, the number of simple n-lacings turns out to be surprisingly complicated, involving all five roots of a particular quintic polynomial. Lists of actual values for the numbers of dense, simple, straight, and general n-lacings are given for $n = 2, \ldots, 8$. You will be interested to know that if you have a boot with eight pairs of eyelets, you have a total of 52,733,721,600 choices as to how you might lace it. The determination of the number of various subcategories of straight lacings relies on the results on one-column lacings from the previous chapter.

In Chapter 4, previous results by other authors (yes, this is not the first work on the mathematics of shoelaces) are extended to find the shortest n-lacings for the various subcategories. So if you mistakenly buy a shoelace intended for a shoe with fewer eyelets than yours, all is not lost. It turns out that the bowtie lacing is the shortest lacing of all, the crisscross lacing is the shortest dense lacing, and the star lacing is the shortest dense straight lacing.

The fifth chapter considers variations on the shortest lacing problem. What if we allow "open" lacings, where one of the segments making up the closed path is removed? What if we allow the two columns of eyelets to be offset from one another? What if we no longer assume the two lines of eyelets are parallel? What if we allow multiple laces on a single shoe?



Types of Lacings

dense: no vertical segments
simple: no backtracking
straight: contains all horizontals
superstraight: all segments vertical or horizontal

Illustration from the book. Courtesy of Burkard Polster.

Chapter 6 is a consideration of the longest n-lacings in the various categories. It turns out, for instance, that the zigzag lacings are the longest of the simple lacings.

In the seventh chapter, the author turns to the question of which lacings are the strongest. By this, he means those lacings that create the greatest horizontal tension, pulling the two sides of the shoe together. It turns out that the two most common lacings, the crisscross for smaller values of the stretch h and the zigzag for larger values of h, are the strongest lacings for the general category, perhaps partially explaining their popularity.

As a mathematician, the author couldn't help but then turn to the question of finding the weakest lacings, which is addressed in the final chapter. He gives a variety of results and conjectures for the various categories.

Will this book do as well as the necktie book? No. That book was aimed at a very general audience, and the minimal mathematics was relegated to the last few pages in order to avoid frightening those potential readers.

The Shoelace Book is aimed at a mathematically inclined audience. The author includes a chunk of mathematics here. None of it is heavy duty, and almost all of it is self-contained, but it does require a certain level of mathematical maturity.

What level? Well, you are reading a review of a math book in the *Notices*, so I am guessing you have the requisite mathematical background. And any undergraduate math major should have the sophistication necessary to appreciate it. It is even possible that this book could be used as a text in a seminar format with math majors, although the lack of exercises would make this a substantial amount of work for the professor. And woe is the administrator who has to deal with irate parents and/or trustees when confronted with the fact that there is a course in their college catalog that teaches the students how to lace their shoes.

There are attempts made to make the book palatable to a wider audience. Witness the inclusion of a variety of comic strips that touch on shoelaces. But the impression they give is exactly that, an attempt to lighten the mathematics that appears. In fact, the strips are not relevant to the material in any chapter and give the impression of someone, author or editor, who went to an online cartoon bank and typed in the words "shoe lace".

But for the mathematically inclined, this is a fun book. The questions are easily stated, and some of the solutions are surprisingly complicated. And there remain plenty of open questions and conjectures in this nascent field of shoelace mathematics. This book will forever change the way you look at your shoes. And it does get you to thinking about the mathematics in the other everyday objects around you.

At the end of the book, there are two appendices. The first discusses related mathematics. This includes how the length of a shoelace is in fact a version of the traveling salesman problem. Then the so-called "shoelace formula" for computing the area of a simple closed polygon in the plane is presented. This is in fact related by analogy only.

The second appendix discusses a variety of "loose ends", including how we tie the two ends of our shoelace together. There is a minor error here, where Polster explains a method for tying your shoes that will hold together better than the traditional way, which he says is also used for tying sutures during surgery. In fact, the description given, and the knot shown (see the last figure in the book) cannot be tied in the ends of a lace or as a suture. The two loose ends of the shoelace to be tied are in fact connected in his particular knot. So do not take Polster's advice and attempt to teach your children this one, or they may end up switching to Velcro.

In this section of the appendix, Polster also explains that when one uses the standard method of tying a shoelace, one creates a knot that is fundamentally either a granny or a square knot. The granny knot is the mark of a novice knot tier, the knot that is well known by sailors not to hold together nearly as well as the square knot. But Polster notes that many of us, perhaps the majority, tie our shoes using the granny knot rather than the square knot. Try this experiment. Tie your shoe. Then stick your fingers in the loops and pull the loose ends through, so you are left with a knot. Is it the granny or the square knot?

I was shocked and chagrined to learn I was a granny tier. No wonder my shoelaces have been coming undone all these years. I just imagined all that wasted time, and the constant distraction of my loose laces, usually occurring in the middle of some crucial thought. Who knows how many times I was on the verge of coming up with the idea of disseminating fun math through shoelaces, when an untied shoelace distracted me from my thought process? I guess my shoelaces were trying to tell me something. Thank goodness, Burkard Polster understood what his shoelaces were trying to tell him.

And of course, as he points out, erroneous lace tying is easily corrected. We can all benefit from the instruction.