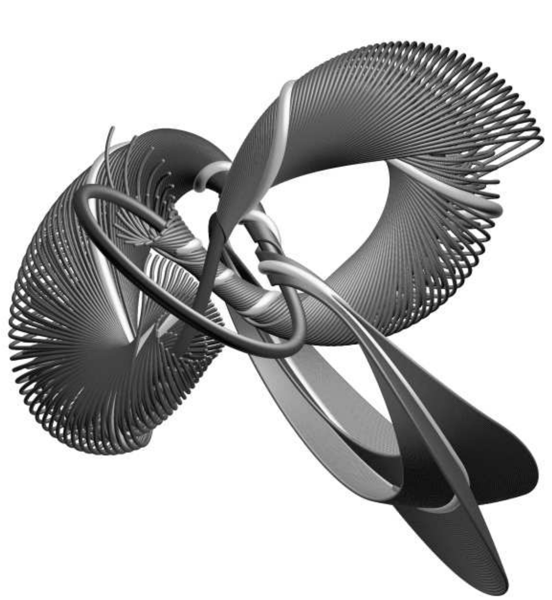


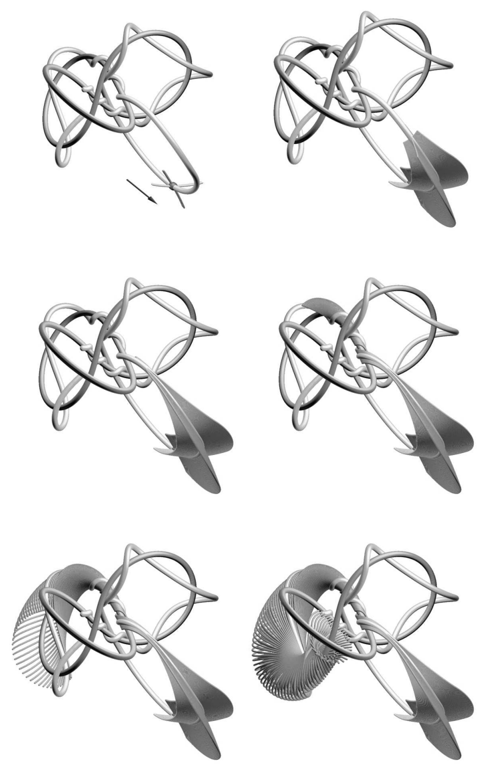
Keep in mind here that the three-sphere may be identified through stereographic projection with $\mathbb{R}^3 \cup \{\infty\}$.

Through each point in the complement there exist several interesting curves. One is its orbit O with respect to the group of diagonal matrices, shown on the cover in white. Others are the stable and unstable manifolds at that point, which in this case correspond to the subgroups of upper and lower unipotent matrices. The flow compresses the stable points into O but uncompresses the unstable ones into a surface that winds chaotically in space around it. The past and future of these trajectories are shown on the cover, one in green and the other in gold. The figures opposite exhibit the forward flow of parts of the stable and unstable curves through one point, as well as the orbit O. (The motion is initially towards the reader). This can give only a feeble idea of what's in the animations of Ghys and Leys.

About the Cover

The cover for this issue was produced by Étienne Ghys and Jos Leys. It shows the trajectories of the stable and unstable curves at a point with respect to the Anosov flow due to the one-parameter subgroup of diagonal matrices of $SL_2(\mathbb{R})$ acting on $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$. As the figure illustrates, the flow has chaotic aspects. Many more—indeed, an astonishing abundance of—such pictures are part of the remarkable article "Lorenz and modular flows: A visual introduction", which Ghys and Leys contributed as the November 2006 installment of the Feature Column on the AMS website `http://www.ams.org/featurecolumn`.

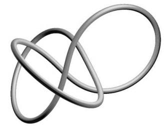
This picture is in fact a still shot from one of a collection of QuickTime^{TM} animations embedded in the article which demonstrate better than any number of still shots the nature of the flow. The article derives from Ghys's presentation at the ICM in Madrid last summer and represents a very promising development in mathematical exposition. The beautiful result of Ghys's work was the coincidence, hitherto unsuspected, of two families of knots: one arising as periodic trajectories in the Anosov flow on $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ pictured here, and the other arising similarly in the flow of the well-known Lorenz attractor.



The development of stable and unstable manifolds (scan across and down).

Hyperbolic Flow on the Space of Unit Lattices

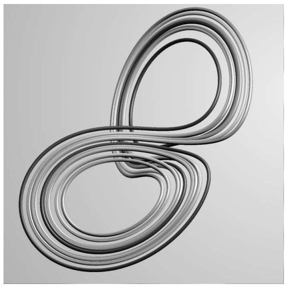
The quotient $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$ may be identified with the space of unit lattices L in $\mathbb{C}$, and via the classical Weierstrass functions $g_2(L)$ and $g_3(L)$ this may in turn be identified with the complement of the discriminant locus $\Delta = g_2^3 - 27g_3^2 = 0$ in the unit three-sphere in $\mathbb{C}^2$, which happens to be a trefoil knot.



The trefoil discriminant locus.

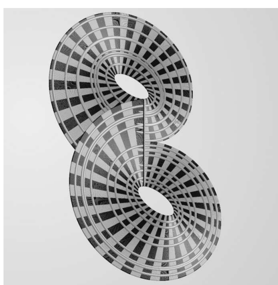
The Lorenz Flow

In an impressive paper published in 1983, Joan Birman and Bob Williams discussed the problem of classifying the knots that appeared as periodic orbits in the attractor associated to the Lorenz equation, the one discovered by Edward Lorenz much earlier as a model for deterministic chaos.



Knots in the Lorenz attractor.

They did carry out a classification, but not quite for the original problem. Instead, they used a geometric model proposed earlier by John Guckenheimer and Williams that transformed the problem into one involving two-dimensional flow on a branched surface and thence into one involving intersections with a slice, hence symbolic dynamics.



Intersecting the transverse slice on a branched surface.

They left open, albeit with some evidence to make their claim plausible, the very difficult problem of justifying this transformation. Recently, Warwick Tucker has shown by machine computation involving interval arithmetic that their model was justified.

Bob Williams commented, "Our paper was essentially the first one that treated periodic orbits as knots. 'Essentially' because … my then colleague, John Franks, had written a paper connecting the Alexander polynomial of knots to a Weil type zeta function that a lot of us were thinking about at the time. … I had given a talk in a seminar on turbulence in Berkeley in 1976 in which I computed one of the zeta functions for the Lorenz periodic orbits. I concluded with the statement that most of them were knotted.

"At this time many dynamicists were thinking about knots—in particular when Vaughan Jones had his breakthrough. … I think the Birman-Williams paper definitely had more impact in dynamics than it did in knot theory."

Joan Birman added, "Our paper was almost ignored by knot theorists. … In some sense the work of Ghys proves that Lorenz made an incredible discovery when he found those equations. They are, in a very real way, the simplest example you can find of the onset of chaos."

References

[1] Joan S. Birman and Robert F. Williams, Knotted periodic orbits and dynamical systems—I. Lorenz's equation, *Topology* **22** (1983), 47–82.

[2] Étienne Ghys, Knots and dynamics, to appear in the proceedings of the ICM in Madrid, 2006. More, equally striking, images on this and other mathematical topics can be found at `http://www.josleys.com/`.

[3] Warwick Tucker, A rigorous ODE solver and Smale's 14th problem, *Foundations of Computational Mathematics* **2** (2002), 53–117. Also available at `http://www.math.uu.se/~warwick/main/rodes.html`. A more informal note by Tucker is the short "The Lorenz attractor exists—an auto-validated proof", at `http://www.cs.utep.edu/interval-comp/interval.02/tuck.pdf`. An informal account of this by Ian Stewart can be found in "The Lorenz attractor exists", *Nature* **406** (2000), 948–9; and another by Marcelo Viana in "What's new on Lorenz strange attractors?", *Mathematical Intelligencer* **22** (2000), 6–19.

All of the figures in this article are due to joint work by Ghys and Leys, but I wish to thank Leys especially for his tireless efforts in producing graphics files on short notice. I wish to thank both of them also for a great deal of patience in preparing the AMS feature column as well as for helping me to write this. Leys used the programs `Knotplot`, `Povray`, *and* `Ultrafractal` *as his graphics tools. Of the last he says, "UF is probably the best program around for drawing fractal patterns. It comes with an extensive library of fractal formulas, which is constantly being extended, since UF allows users to write their own formulas and publish them for general use. It was this scripting feature that I found so appealing, as it opens the door for possibilities beyond fractals into the representation of geometric objects in general. … UF is very fast—the final image of one of our animations, with its* 2,400,000 *spheres takes about 45 seconds to complete on screen. In that time it calculates three Jacobi* θ *functions* 2.4 *million times!"*

—Bill Casselman, Graphics Editor
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