Book Review

Fearless Symmetry

Reviewed by Dino Lorenzini

Fearless Symmetry: Exposing the Hidden Patterns of Numbers
Avner Ash and Robert Gross
Princeton University Press, 2006
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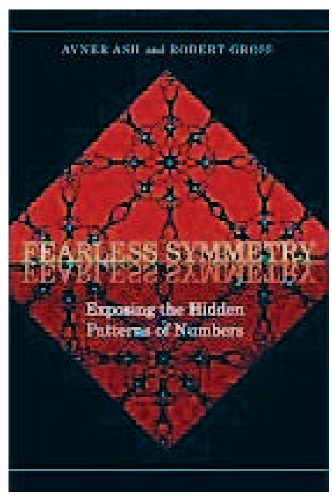
Every professional mathematician who has ever attended a cocktail party has had to answer the question: what is research in mathematics? For most of us, there is no easy answer, especially when the person asking adds, as an afterthought, "I thought that everything was known by now in mathematics." The professional mathematician then gauges the length of time that he or she has before the querier becomes sorry he asked, and then proceeds to discuss in down-to-earth terms a catchy mathematical subject. For number theorists, public key cryptography or Fermat's Last Theorem often has the potential to interest the casual listener.

Ash and Gross, in this welcome book, answer the question in the way we would all like to have the opportunity to do: at a leisurely pace, with motivating examples, and with digressions on how mathematicians really think, and how mathematics is "made". They chose a central, easy-to-state mathematical subject—equations and how to solve them—to motivate the mathematical adventure they are proposing to their readers. Starting from scratch, they explain some fundamental tools developed by mathematicians to tackle this motivating problem and many others. Their ultimate, fearless, goal is to explain the concept of reciprocity law for a representation of the Galois group of $\mathbb{Q}$, culminating in the use of such a reciprocity law in Andrew Wiles' proof of Fermat's Last Theorem. In this, they more than succeed.



Fearless Symmetry is written to be accessible to a broad audience, centered principally on those who have studied calculus. This is not a textbook, and very few proofs are offered. On the other hand, concrete examples are discussed, motivating the given definitions.

The first part of the book introduces groups, representations, complex numbers, modular arithmetic, and quadratic reciprocity. This could serve as a primer for a first course in abstract algebra. Undergraduates learning such topics for the first time would benefit from reading this book as a complement to a course, as this book makes for a relaxed introduction to the concepts and provides an interesting overview of where these may lead.

The second part of the book introduces the basics of Galois theory, elliptic curves, algebraic integers, Frobenius elements, and the far-reaching link that representations of the Galois group of $\mathbb{Q}$ can

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be obtained from geometric objects such as elliptic curves.

The third part of the book is the culmination of this adventure. Reciprocity laws are introduced and succinctly described by the authors as “the bringing together of two patterns. One pattern is the set of traces of Frobenius elements acting in a Galois representation. The other pattern comes from the black box—another mathematical object of some different type” (page 235). In the work of Wiles, the black box is analytic in nature, as the pattern is the set of coefficients in a Taylor expansion of a modular form. This is by no means an easy subject, but the reader is well-rewarded for her effort when seeing these tools applied to concrete mathematical problems such as the description of all rational solutions to a given equation. That it took over 300 years to prove that the integer solutions to Fermat’s equation $x^n + y^n = z^n, n > 2$, consist only in the trivial ones is a testimony to the difficulty of the field. The authors have taken the challenge and have succeeded admirably in bringing the essence of these cutting-edge research topics to a lay audience.