Book Review

Symmetry and the Monster, One of the Greatest Quests of Mathematics

Reviewed by Robert L. Griess Jr.

Symmetry and the Monster, One of the Greatest Quests of Mathematics
Mark Ronan
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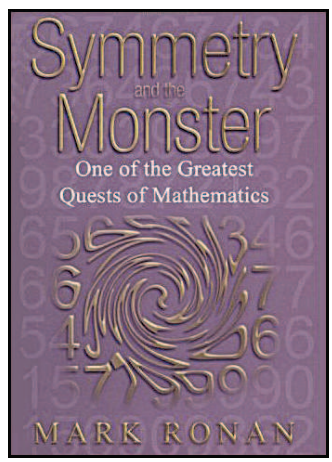
A new book about simple groups has appeared, this one for general audiences. The title refers to the wide role of groups as symmetries in mathematics and science, as well as to the "monster", a particular large finite simple group that has gotten much attention during the last three decades. The author, Mark Ronan, does research in groups and geometries.

This book tells a history of simple groups, mainly continuous groups (Lie groups) and finite simple groups, in a nontechnical way. (Many kinds of groups, simple and otherwise, constructed in combinatorial group theory, arithmetic, topology, etc., are not considered.) Historical figures in the story are described (Niels Abel, Évariste Galois, Wilhelm Killing, Felix Klein, Friedrich Engel, William Burnside, . . .). There are lengthy accounts of their personal and professional dramas. The effects of the world wars and major societal changes on mathematics are discussed. Group theorists from recent decades, many of whom are still active, are quoted. The author gives an account of connections between continuous and finite simple



groups and other areas of mathematics and theoretical physics, notably the significant role of group theory in physics and the moonshine phenomena.

The "monster" of the title plays an unusually interesting role in the big story. The text contains descriptions of many of the twenty-six so-called "sporadic simple groups", those finite simple groups that do not lie in the infinite families of alternating groups or groups of Lie type. The five Mathieu groups were discovered in the nineteenth century, and the following twenty-one were discovered around 1965–1975. The largest of these sporadic groups is the monster, discovered in 1973.

The title is a bit hard to interpret. The monster is treated as the conclusion of a long journey. In a really unique way, the monster embodies important and mysterious connections between areas of mathematics and physics, and much more of this could be discovered in the future. A more cautious view is that the monster is one actor among many (obviously, an important one) in the ongoing story of groups in mathematics and science. I did not find the "quest" defined. I suppose it is a quest for the simple groups themselves and for fuller understanding of groups in the scientific universe.

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The book is pitched towards the scientifically aware general audience and is appropriate for young people. It gives a satisfying account of how the story of Lie groups and finite simple groups evolved, both as a human struggle to understand the universe and as an introduction to the theories. The reader gets a taste of technicalities. One sees both the common example of a dihedral group of order 8 (p. 48) as well as the exotic-looking special calculation (p. 176)

$$1+3{,}968{,}055+23{,}113{,}728+2{,}370{,}830{,}336+$$
$$11{,}174{,}042{,}880=13{,}571{,}955{,}000$$

of the number of $2A$-involutions in the "baby monster", a simple group discovered by Bernd Fischer in 1973.

The visuals are thoughtfully done (e.g., rigid symmetries of Platonic solids in Chapter 1 and automorphisms of graphs and geometries in Chapter 9). There are helpful tables, such as the "periodic table" of Lie groups (p. 66) and all sporadic groups (called "The 26 Exceptions") in Appendix 4, p. 244.

Stories of mathematicians through the early twentieth century are adapted from other authors. This author adds new material from published quotations and his own interviews with mathematicians. This single book covers two hundred years of group theory, essentially the entire relevant historical interval. The narrative moves briskly. I found myself eager to turn the pages and move on to the next chapter. The author is articulate and chooses well.

The general reader will likely appreciate the book's many side remarks on such topics as the Nobel and Abel Prizes, the Fields Medals, and the Oberwolfach Forschungsinstitut. The book also relates anecdotes about mathematicians the author knew personally.

The classifications of Lie groups and of finite simple groups are treated, but rather lightly. These two classifications have very different natures.

The Contents

A Quick Tour

The author treats many topics. We shall devote more space to the recent ones, since they have generally received less attention.

The text begins with Theaetetus and the story of the Platonic solids. The dramatic story of the young and brilliant Évariste Galois is discussed in detail, and there is a generous exposition on solving low-degree polynomial equations. From degree 5 and up, one is unable to solve polynomials with formulas involving only radicals and rational operations. Finite simple groups are lurking!

The story of Sophus Lie is colorful, and his interactions with Felix Klein, Wilhelm Killing, and Élie Cartan are quite significant and fascinating. Lie algebras and Lie groups have deeply affected mathematics for over a century and will continue to do so for a long time.

The chapter on Lie groups and physics gives a brief account of relativity, special relativity, and subatomic particles. It notes that electrons are limited to lie in discrete states and that these correspond to properties of the Lie groups.

The contributions of Leonard Eugene Dickson to finite simple groups are enormous and not widely appreciated. He created finite field analogues of the exceptional Lie groups of types G_2, F_4, and E_6. He was a fantastically productive worker in algebra and number theory. Dickson's early book [4] contains a wealth of information about many finite groups.

In the 1950s Claude Chevalley gave a uniform procedure for defining analogues of Lie groups over any field, a beautiful and very important broadening of Lie theory [3]. These "Chevalley groups" included the families of groups created by Dickson and placed them in a context.

The book offers an account of the Bourbaki group and how the world wars affected mathematics and the immigration of mathematicians to North America, especially European Jews.

The chapter title "The Man from Uccle" is both a reference to Jacques Tits's hometown Uccle, Belgium, and a pun on a popular 1960s television show. The author gives Tits special status as a great architect of geometric theories (covering Lie groups, arithmetic groups, finite groups, and other algebraic systems) and as an insightful group theorist. Tits's theory of buildings includes his important characterization of groups of Lie type by geometries.

The next chapter, "The Big Theorem", is about the Odd Order Theorem of Walter Feit and John Thompson (that all finite groups of odd order are solvable) and its context, the early thinking about the classification of finite simple groups. Thompson is considered the great pathbreaker in modern finite group theory, starting with his graduate work in the late 1950s and continuing for years of profoundly influential results. In 1970 Thompson was awarded a Fields Medal for work on N-groups (finite groups in which local subgroups are solvable). Richard Brauer's significant role and life are sketched. Michio Suzuki's contributions are mentioned in a few places in the book. I feel that Suzuki's early work in the classification could have been emphasized more in this chapter (his later works on simple groups are adequately reported). Finally, one reads about the surprise of Zvonimir Janko's finite simple group of order 176,560, the first sporadic group discovered in about a century [13].

"Pandora's Box" gives an introduction to groups and finite geometries, Mathieu groups, and the ideas in the mid-1960s that led to discoveries of many other sporadic groups.

Algebraist Reinhold Baer had a broad view of mathematics and was quite supportive of young talent. He had particular influence on the young Bernd Fischer, who pursued rather personal interests about algebraic systems (great success followed years later). The influence of Reinhold Baer on finite groups seems not to be widely known. Dieter Held, Zvonimir Janko, John Thompson, Jacques Tits, and others attended his seminars. Another account of Baer's career is [7].

Zvonimir Janko, like Fischer, had very strong personal ideas about where to look for new simple groups, and he worked them quite hard. Janko's most successful theme was the so-called "O_2 extraspecial" hypothesis for centralizers of involutions. The flavor of Janko's program was more "internal group theory" rather than "external geometry". Fischer and Janko each found several new sporadic groups but by mining rather different parts of the group theory terrain.

The chapter on the Leech lattice explains the concepts of sphere packing, lattices, and the exceptional 24-dimensional packing discovered by John Leech. It reports the amazing half-day in the life of John Conway when he understood the Leech lattice and its isometry group and saw how to explain their remarkable treasures. (In Appendix 3, p. 242, the author lists all minimal vectors of the Leech lattice, a pretty counting exercise.)

The chapter "Fischer's Monsters" starts with the basic theory of dihedral groups, which are finite groups generated by pairs of distinct involutions, and how they arise as linear transformations on high-dimensional vector spaces. Bernd Fischer explored a very natural class of groups generated by involutions and found new sporadic groups. The central hypothesis is simply stated: a conjugacy class of involutions is given and any pair of involutions from that class either commutes or generates a dihedral group of order 6. This property is held by the transpositions in symmetric groups and by particular conjugacy classes in classical matrix groups in characteristics 2 and 3. It is still so amazing to me that this simple and geometrically natural property leads through familiar examples to three new sporadic groups, Fi_{22}, Fi_{23}, and Fi_{24} (the first two are simple and the third has a simple commutator subgroup Fi_{24}' of index 2). Michael Aschbacher, Franz Timmesfeld, and Bernd Fischer relaxed these conditions to analyze wider classes of groups. Developing this theme further, Fischer found the evidence for the "baby monster" and (a bit later) the "monster".

The chapter on the Atlas is about the collaborative effort to compile, check, and refine data on simple groups such as their character tables, maximal subgroups, and presentations. It is interwoven with a brief account of the program to classify finite simple groups, emphasizing the 1970s decade. This reviewer feels that the relationship between these programs may not be clear in the text. The effort to classify finite simple groups is far more vast than the program to compile the Atlas and indeed enabled that program.

The team classification effort was led by the energetic and visionary Daniel Gorenstein. This book's account of Gorenstein's amazing life force is truthful. I am happy that his fantastic dedication and organizational skills are reported here in some detail; they were significant to the classification team effort and to the Rutgers University mathematics department. Michael Aschbacher was the younger leader of this team, due to his aggressive new techniques and sustained rapid output. A feeling developed only in the late 1970s that a classification might finally become reality. It would depend on solving a few specific problems, and this was indeed clarified by a theorem of Gorenstein and Richard Lyons. In the early 1980s (possibly at the AMS meeting, January 1981), Gorenstein announced that the classification was done. At first, that declaration was generally supported by the classification theorists. However, it became increasingly difficult to defend as the decade of the 1980s went by.

The completeness claim created a serious dilemma for me personally. On one hand, I had great respect for the top finite group theorists and the impressive classification machinery. However, such a huge claim seemed rushed to me. If the classification were indeed complete, it would be not only the longest proof in history but also an *induction* proof. The expected list of finite simple groups does not yet correspond exactly to anything else in mathematics, and we were aware that past technical mistakes had incorrectly ruled out groups later found to exist. Not all relevant manuscripts had been turned in at that time, and there were many separate efforts in progress.

Since the declaration, gaps were found and treated, and improvements were achieved. Especially noteworthy is the recent [1] (two volumes!), in which Michael Aschbacher and Stephen Smith finally settle the "quasithin" problem. A solution was claimed in the mid-1980s by another mathematician who ultimately did not publish his long manuscript. Another example is the 1989 article [10], which finally proved a necessary classification result. The person who originally claimed this result in the early 1980s never published a proof.

Revision continues, led principally by Richard Lyons and Ronald Solomon. The wait for resolution is justified, given the extreme length and difficulty of the material (thousands of journal pages) and the importance of knowing the finite simple groups. No new finite simple group has been found since 1975.

The chapter "A Monstrous Mystery" shows how surprising connections ("moonshine") were discovered between the monster and a set of genus 0 function fields which correspond to the group of 2- by-2 integral matrices of determinant 1 and some closely related matrix groups. The latter topic has been well studied since the nineteenth century. These connections, discovered by John McKay and John Thompson (late 1970s) and enriched by John Conway and Simon Norton, were truly a surprise and to this day remain a challenge to explain. Thompson conjectured the existence of a graded representation of the monster whose graded traces give the Hauptmoduln, certain normalized generating series for the genus 0 function fields mentioned above.

The chapter "Construction" is about the work of Robert Griess (this reviewer). In 1973 Fischer and Griess independently produced group theoretic evidence that there could be a sporadic simple group with the order $2^{46}3^{20}5^{9}7^{6}11^{2}13^{3}17\cdot 19\cdot 23\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71 \sim 8.08\times 10^{53}$ and additional specific properties. Their evidence was the passage of many local tests by the hypotheses, but there was at that time no existence proof. Furthermore, because such a putative group would be so large and lack any small dimensional representation, the possibility of construction seemed doubtful, even with computers. In late 1979 Griess returned to thoughts of a possible existence proof. He constructed such a group as matrices in dimension 196883 over the rational numbers and announced the results by mail on January 14, 1980. It was quite important that this group leave invariant an algebra structure (a product) on the 196883-dimensional vector space, because such a requirement gave useful constraints for making definitions (earlier work of Norton indicated that if an irreducible 196883-dimensional representation exists, then there should exist some invariant algebra structure). Finally, Griess defined a particular product on the space and constructed enough symmetries of it to generate a large finite simple group. This was done with abstract group theory, "by hand" as it were, by labeling the 196883-space with data from the famous Leech lattice and its ambient 24-dimensional vector space. Corollaries of the construction included new and easy existence proofs of many previously known sporadic groups. Twenty of the sporadic groups were unified, in a sense.

The final chapter introduces some ideas from physics and sketches their connections to simple group theory. Richard Borcherds made two important contributions here. First, he showed the relevance of vertex algebras and how to work with them. The hyperbolic geometry that is important in working with space and time is useful for explaining lattices, vertex algebras, and sporadic groups. Borcherds furthermore verified the basic conjecture of Thompson, that the moonshine vertex operator algebra of Igor Frenkel, James Lepowsky, and Arne Meurman does afford graded traces, which are the relevant Hauptmoduln. In 1998 Richard Borcherds won a Fields Medal for his work on moonshine.

What the Book Gives Us

Ronan's real achievements in this popular science book are (1) gathering testimony about modern work on finite simple groups and providing expositions of the new ideas that arose, (2) integrating the older and newer parts of the group theory history in a single narrative, (3) revealing the human sides to the story and how they mesh with the scientific.

The mathematics community is generally aware of certain highlights of modern finite group theory such as Brauer's proposals (mid-1950s) to start the classification program, the breakthroughs of John Thompson, the Feit-Thompson Odd Order Theorem, discoveries of sporadic groups, and the announcement by Gorenstein that finite simple groups were classified. Many human aspects of the story are not widely known.

The material from recent times is especially welcome. This history could be lost within decades.

Notations and Terminology

The author never seems to give the definition of a group, as one would see in a text or college course. Certain standard mathematical concepts are given new names. The author insists on the term "atoms" rather than simple groups. He uses the term "cross section" for centralizer of involutions (a footnote offered a definition, but I was disappointed). There is also an avoidance of subscripts. The text uses $A2, D5, \ldots$ for the Lie groups of respective type $A_2, D_5, \ldots$ and $M22$ for the Mathieu group M_{22}. Could this be to make things easier for the reader? On the other hand, one finds superscripts (M^{22}, etc.; p. 178).

I find these "simplifications" not only unnecessary but probably counterproductive. As a youngster, I enjoyed reading popularizations of science and mathematics (George Gamow's *One, Two, Three... Infinity* is a delightful example). Such friendly guidance gave access to the proper names of the scientific objects and concepts. To me it meant being treated as an adult and valued as a potential scientist, even though I felt overwhelmed by the great ideas.

Comments and Corrections

Marshall Hall independently discovered, constructed, and proved uniqueness of the simple group of order 604,800 [12] around the same time as Janko made his discovery [14]; each refers to the work of the other. Joint credit is appropriate. Most of the time I use the notation HJ instead of J_2; both notations are used by the group theory community.

"John McLaughlin" should be Jack McLaughlin.

I disagree with parts of the author's account about the character table of the monster (it appears in [2]), whose determination presented unusual difficulties. This character table was calculated in the late-1970s [15], *based on the reasonable hypothesis* that there was an irreducible character of degree 196883 (a conjecture of mine [8]). Existence of such a character followed from the later works [9, 10]. As far as I know, there is no other existence proof of such a character, and there is no character table determination that avoids starting with an irreducible of degree 196883.

The "Notes" section was enjoyable. The author used a lot of recent scholarship.

After the four appendices comes the glossary. Some of its "definitions" seem vague (e.g., j-function, Lie group), and there are terms (cross section, deconstruction, periodic table) that are not generally used the way the author does. The item on "group" is disappointing (there is no reference to the associative law). The meaning suggested by the definition of "periodic table" is misleading, because the families of finite simple groups of Lie type in positive characteristic are not in one-to-one correspondence with the families of Lie groups (see [2, 11, 5, 6]; there are nearly twenty families, not seven as the author says). A complete list would take two pages.

The book makes brief mention of group character theory in connection with Burnside from a century ago, but there is no discussion of its important role in the classification of finite simple groups, especially around the 1950s and 1960s (for example, early centralizer of involution problems and characterizations by Sylow 2-subgroups).

Summary

The book is quite accessible, artfully written, and rich in specifics, and it stresses the human side of the drama. Though I have been a long-time participant in the story, I found myself learning much in every chapter and not wanting to put the book down.

Ronan's book is not the unique way to tell the story. It reflects his tastes (e.g., emphasis on buildings). My own version would tell more about methods that led to new sporadic simple groups (meaning methods not directly inspired by Lie theory, such as centralizer of involution studies and explorations of rank 3 graphs). Many ideas which drove finite simple group theory are best understood as part of the evolution of *general finite group theory*. Also, I would describe the group theory community around Chicago and Champaign-Urbana in the late 1960s: Jon Alperin, Helmut Bender, Norman Blackburn, Everett Dade, Len Evens, Paul Fong, George Glauberman, David Goldschmidt, Christof Hering, Morton Harris, Marty Isaacs, Noboru Ito, Bill Kantor, Richard Lyons, Len Scott, Gary Seitz, Michio Suzuki, John Thompson, John Walter, Warren Wong.... The heavy schedule of seminars, the many visitors, and the intellectual ferment were great catalysts to the development of general finite group theory. This includes representation theory, group cohomology, local group theory, geometry, and permutation groups, not just simple groups, the main topic of this book. I was fortunate to have been there and learned so much. It was a great time.

The author gives a nuanced account of the research process, describing both the scientific content and the psychological aspects. He describes learning from the masters, the loneliness of doing groundbreaking research, collaborations both harmonious and strained, and the sometimes difficult path to ultimate recognition. The many examples in the book should be informative, especially for young readers. They can help develop survival skills and attitudes for success. The world of mathematicians is competitive.

Symmetry and the Monster, One of the Greatest Quests of Mathematics is rewarding and recommended for established scientists and young people.

Acknowledgment

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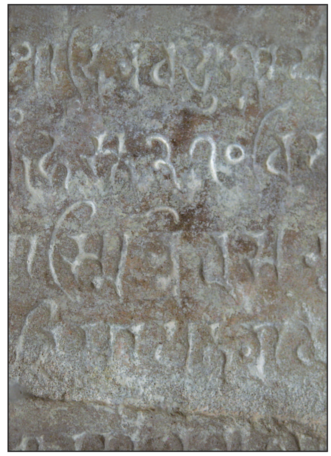
About the Cover

The zero of Gwalior



The cover displays a photograph of what may be the oldest extant decimal '0' in India, where decimal place notation was invented. It was suggested by the opening chapter of Constance Reid's well known book *From zero to infinity*, which is reviewed in this issue on the occasion of a new printing. The number '270' visible in the photograph is found on a tablet on the wall of a small temple on the eastern approach to the medieval fortress of Gwalior, a city in the north of the state of Madhya Pradesh. The temple itself was erected in 876 A. D., and inside it is a tablet commemorating the donation of land, of dimension 270 *hastas* by 187 *hastas* and other gifts to another local temple, as well as a daily gift of 50 garlands of flowers to the two temples. All these numbers appear in decimal place notation, in more or less recognizable form.

There are plausible arguments that full decimal place notation for integers was invented in India some time in the 6th century. It seems at least clear from this tablet that it had come into common usage by the mid 9th century. The Babylonians, followed by the the Alexandrian Greeks, had used a symbol for zero in astronomical calculations with sexagesimal notation, and it seems almost certain that this usage was imported into India sometime in the years 200–400 A. D. To what extent this influenced the invention of decimal place notation is not clear.

Another candidate for the oldest extant zero is the Bakhshali Manuscript, now in the Bodleian Library of Oxford University, but it has not been dated, except on uncertain if reasonable paleographical grounds. In this manuscript can be found the manipulation of rational numbers with a facility equal to our own and an enthusiasm that exceeds the average of modern times—at one point in the course of a calculation involving the square root of a non-square integer can be found the rational number

$$\frac{50753383762746743271936}{7250483394675000000}\,!$$

(The calculation yielding this has been reconstructed by Takao Hayaishi in his treatment of folio 46 *recto* in his impressively thorough edition of the MS.)

I wish to thank Renu Jain of the mathematics department of Gwalior University, as well as A. K. Singh of the archaeology department, for acting as guides in Gwalior.

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