The Mathematical Work of the 2006 Fields Medalists

The *Notices* solicited the following articles about the works of the four individuals to whom Fields Medals were awarded at the International Congress of Mathematicians in Madrid, Spain, in August 2006. (Grigory Perelman was awarded the medal but declined to accept it.) The International Mathematical Union also issued news releases about the medalists' work, and these appeared in the October 2006 *Notices*.

—*Allyn Jackson*

The Work of Andrei Okounkov

*Nicolai Reshetikhin**

Perhaps two basic words that can characterize the style of Andrei Okounkov are clarity and vision.

His research is focused on problems that are at the junction of several areas of mathematics and mathematical physics. If one chooses randomly one of his papers, it will almost certainly involve more than one subject and very likely will have a solution to a problem from one area of mathematics by techniques from another area. Many of his papers opened up new perspectives on how geometry, representation theory, combinatorics, and probability interact with each other and with other fields.

Typically, one also finds among the results of each of his papers a beautiful explicit formula.

The variety of tools Okounkov uses is very impressive. He has the rather unique quality of moving freely from analysis and combinatorics to algebraic geometry, numerical computations, and representation theory.

Let me focus on some representative examples.

One of his remarkable results is the Gromov-Witten and Donaldson-Thomas correspondence, which is an identification of two geometric enumerative theories. These results are intrinsically related to his works on Gromov-Witten invariants for curves, on random matrices, and on dimer models.

Works on Dimers

Dimer configurations are well known in combinatorics as perfect matchings on vertices of a graph where matched vertices are connected by edges. In the 1960s, Kasteleyn and Fisher computed the partition function of a dimer model on a planar graph and the local correlation functions in terms of Pfaffians of the so-called Kasteleyn matrix [3]. Since that time, dimer models have played a prominent role in statistical mechanics.

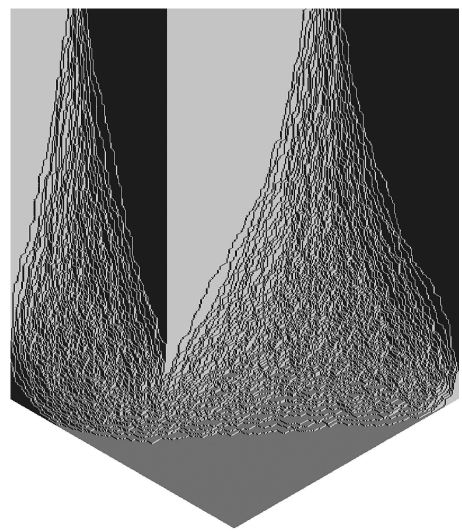

Figure 1. A random surface representing a random dimer configuration on a region of hexagonal lattice.

A pair of dimer configurations on a bipartite planar graph defines a stepped surface that projects bijectively to the graph. Thus, random dimer configurations on bipartite planar graphs can be regarded as random stepped surfaces. A snapshot of such a random surface for dimer configurations on a domain in a hexagonal lattice is shown in Figure 1.

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Looking at this picture, the following is clear: on the scale comparable to the size of the system, such random stepped surfaces are deterministic. This phenomenon is by its nature close to deterministic limits in statistical mechanics (also known as hydrodynamical limits), to the semiclassical limit in quantum mechanics, and to the large deviation phenomenon in probability theory. It is also clear from this snapshot that fluctuations take place on the smaller scale.

The nature of fluctuations changes depending on the point in the limit shape. For example it is clear that the fluctuations in the bulk of the limit shape are quite different from those near a generic point at the boundary of the limit shape, those near singular points at the boundary of the limit shape, or those near the points where the limit shape touches the boundary. These empirical observations are confirmed now and quantified by precise mathematical statements. Okounkov made an essential contribution to these results.

In joint work with Kenyon and Sheffield [4] Okounkov proved that the limit shape for periodically weighted dimers is the graph of the Ronkin function of the spectral curve of the model. A similar description was obtained for local correlation functions in the bulk. This work was based on Kasteleyn's results.

In subsequent papers with Kenyon, Okounkov gave a complete description of real algebraic curves that describe the boundary of the limit shape in a dimer model. It turns out that natural equivalence classes of such curves form the moduli space of Harnack curves [5]. It also turned out that such curves describe a special class of solutions to the complex Burgers equation [6].

Fluctuations near the generic and special points of the boundary are described in [12] and [13]. In this work the dimer model was reformulated in terms of the Schur process and then in terms of vertex operators for gl_∞. Some key ideas used in these papers go back to previous works of Okounkov on representation theory.

All these results are characteristic of the style of Okounkov: the problem in statistical mechanics is solved using tools from algebraic geometry and representation theory. The moduli space of Harnack curves is described in terms of Gibbs measures on dimer configurations. The same space is identified as the space of certain algebraic solutions to the complex Burgers equation.

Gromov-Witten Theory

Gromov-Witten theory studies enumerative geometry of moduli spaces of mappings of algebraic curves (Riemann surfaces) into some fixed algebraic variety.

Let us recall the idea of the Gromov-Witten invariant. Let X be a nonsingular complex projective variety, and let $\overline{M}_{g,n}(X,\beta)$ be the space of isomorphism classes of triples $\{C, p_1; \dots, p_n; f\}$ where C is a complex projective connected nodal curve of genus g with n marked smooth points $p_1, \dots, p_n$ and $f : C \to X$ is a stable mapping such that $[f(C)] = \beta$. Here stability means that components of C that are pre-images of points with respect to the mapping f have finite automorphism groups. Let $\mathrm{ev}_j : (C, p_1; \dots, p_n; f) \to f(p_j)$ be the evaluation mapping. Denote by $ev_j^*\alpha \in H^*(\overline{M}_{g,n}(X,\beta))$ the pull-back of a class $\alpha \in H^*(X)$. Let L_j be a line bundle on $\overline{M}_{g,n}(X,\beta)$ whose fiber over the point $\{C, p_1; \dots, p_n; f\}$ is $T^*_{p_j}C$. The *Gromov-Witten invariants* of X are intersection numbers

$$(1) \quad < \tau_{k_1}(\alpha_1) \dots \tau_{k_n}(\alpha_n) >^X_{\beta,g} = \int_{[\overline{M}_{g,n}(X,\beta)]} \wedge_{j=1}^n c_1(L_j)^{k_j} \mathrm{ev}_j^*(\alpha_j).$$

Here the (virtual) fundamental class $[\overline{M}_{g,n}(X,\beta)]$ was constructed in the works of Behrend-Fantechi and Li-Tian.

When X is a point, Witten [15] conjectured that the generating function

$$(2) \quad f(t_0, t_1, \dots) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k_1+\cdots+k_n=3g-3+n} < \tau_{k_1} \dots \tau_{k_n} >_g \prod_{j=1}^n t_{k_j}$$

is a tau-function for the KdV integrable hierarchy satisfying the additional "string equation". The first proof of this conjecture was given by Kontsevitch in [7]. An alternative derivation of the key formula from [7] was later given by Okounkov and Pandharipande.

When X is a curve, the Gromov-Witten invariants were described completely by Okounkov and Pandharipande [10][11]. They showed that when $X = \mathbb{P}^1$ the generating function for the Gromov-Witten invariants is a tau-function for the Toda hierarchy, again with a special constraint similar to the "string equations". They also showed that the case when X is a point can be obtained by taking a limit of the $X = \mathbb{P}^1$ case.

Another remarkable result by Okounkov and Pandharipande is the following explicit formula for GW-invariants when X is a curve. Let $\beta = d[X]$, and suppose all α_i are equal to ω (the Poincaré dual of a point). Then

$$(3) \quad < \tau_{k_1}(\omega) \dots \tau_{k_n}(\omega) >^X_{d[X],g} = \sum_{|\lambda|=d} \left(\frac{\dim(\lambda)}{d!}\right)^{2-2g} \prod_{j=1} \frac{p_{k_i+1}(\lambda)}{(k_i+1)!}.$$

Here the sum is taken over all partitions of d and

$$(4)\quad p_k(\lambda) = \sum_{j\geq 1}\left((\lambda_j - j + 1/2)^k - (-j+1/2)^k\right) + (1-2^{-k})\zeta(-k).$$

This formula for GW-invariants of curves is rooted in the relation between the GW-invariants and Hurwitz numbers. Recall that the latter are the numbers of branched coverings of X with given ramification type at given points. The branched coverings of X were studied in [1], [2], where the problem was resolved essentially by using representation theory of $S(\infty)$.

Donaldson-Thomas Invariants

Let X be a three-dimensional algebraic variety. Algebraic curves $C \subset X$ of arithmetic genus g with the fundamental class $\beta \in H_2(X)$ are parameterized by the Hilbert scheme $\mathrm{Hilb}(X;\beta,1-g)$. Let $c_2(\gamma)$ be the coefficient of $\gamma \in H^*(X)$ in the Künneth decomposition of the second Chern class of the universal ideal sheaf $\mathcal{J} : \mathrm{Hilb}(X;\beta,\chi)\times X \to X$. The Donaldson-Thomas invariants are:

$$(5)\quad <\gamma_1,\ldots,\gamma_n>_{\beta,X} = \int_{[\mathrm{Hilb}(X;\beta,\chi)]}\prod_{j=1}^{n} c_2(\gamma_j).$$

Here $[\mathrm{Hilb}(X;\beta,\chi)]$ is the (virtual) fundamental class of $\mathrm{Hilb}(X;\beta,\chi)$ constructed by R. Thomas. It is of dimension $-\beta K_X$ where K_X is the fundamental class of X. This is the same dimension as the dimension of $\overline{M}_{g,0}(X,\beta)$.

The conjecture of Maulik, Nekrasov, Okounkov, and Pandharipande [8] relates suitably normalized generating functions of Gromov-Witten invariants for $\overline{M}_{g,0}(X,\beta)$ and Donaldson-Thomas invariants of $\mathrm{Hilb}(X;\beta,\chi)$. This conjecture was proved for local curves and when X is the total space of the canonical bundle of a toric surface.

One can argue that these results were inspired by observations from [14] and from the computation of the Seiberg-Witten potential in [9], and that they can be regarded as a generalization of the formula (3) for Gromov-Witten invariants of a curve. It is also remarkable that combinatorics of partitions came up as a computational tool in this topological subject.

This brief and incomplete description of the achievements of Andrei Okounkov might underestimate one important feature of his work, namely, its unifying quality: Seemingly unrelated subjects have become parts of one field. In this sense his contributions also have an important organizing value.

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The Work of Andrei Okounkov

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In spite of his youth, Andrei Okounkov has gone through several periods in his fruitful creative work: he started in asymptotic representation theory, and then extremely quickly went one by one through several areas, subsequently enlarging his knowledge of a number of subjects in algebraic geometry, symmetric functions, combinatorics, topology, integrable systems, statistical physics, quantum field theory, etc.—modern topics of great interest to all mathematicians. Each change in topic had a prehistory in the previous one and brought useful erudition. The influence of the first period of Okounkov's activity—symmetric functions, asymptotic representation theory, Young diagrams, partitions, and so on—one can clearly feel. I want to say a few words only about his first achievements (and not even about all of them), which are perhaps not so well known and which are right now a little bit in the shadow of his great subsequent successes of the last several years. These subjects also are closer to me because in some sense I started with them.

I will mention several of the first of Okounkov's results: a new proof of Thoma's theorem giving the solution of the problem about admissible representations of $\mathfrak{S}_\infty$; some distinguished formulas; contributions to the asymptotic Plancherel measure on Young tableaux, in particular, the analysis of fluctuations of random Young diagrams and the link with random matrices and matrix problems; and several results on symmetric functions and characters.

Asymptotic Representation Theory: Finite and Infinite Symmetric Groups

The modern theory of the representations of infinite symmetric groups started with the celebrated and fundamental theorem of E. Thoma (1964) about normalized characters of $\mathfrak{S}_\infty$ (the group of all finite permutations of a countable set). It asserts that an indecomposable normalized character (= a positive-definite central function $\chi(.), \chi(\mathbf{1}) = 1$) has the form

$$\chi_{\alpha,\beta}(g) = \prod_{n=2}^{\infty} s_n(\alpha,\beta)^{r_n(g)},$$

where $r_n(g), n > 1$, is the (finite) number of cycles of length n in the permutation $g \in \mathfrak{S}_\infty$; the parameters of the character ("Thoma's simplex") are $\alpha = (\alpha_1, \dots)$ and $\beta = (\beta_1, \dots)$, with $\alpha_1 \geq \alpha_2 \geq \cdots \geq 0$ and $\beta_1 \geq \beta_2 \geq \cdots \geq 0$. The sums $\sum_{k=1}^{\infty} \alpha_k + \beta_k \leq 1$ and $s_n(\alpha,\beta) = \sum_{k=1}^{\infty}[\alpha_k^n + (-1)^{n-1}\beta_k^n], n > 1$, are supernewtonian sums. The characters are multiplicative with respect to decomposition of the permutation on the cycles, and the value on a cycle of length n of the previous character is $s_n(\alpha,\beta)$.

Thoma's proof was based on hard analysis and the theory of positive-definite functions. It had nothing in common with proper representation theory. In the 1970s the author with S. Kerov gave a "representation-theoretic" proof of the theorem in the framework of the general approach to the approximations of those characters with the irreducible characters of finite symmetric groups, some ergodic ideas, and an important interpretation of those parameters α, β above as frequencies of the rows and columns in growing random Young tableaux. Then, in 1995, Okounkov suggested in his thesis a completely new idea and gave a third proof, based on a nice analysis of the elementary operators in the space of the representation. This proof immediately gives a new interpretation of the parameters as eigenvalues of some operators. Moreover, his goal was the solution of a more general problem that had been posed by his advisor G. Olshansky, namely, the description of all so-called admissible representations of the *bisymmetric* group (which includes the problem about the characters above). This is a problem about the complete list of the irreducible representations of the group $\mathfrak{S}_\infty \times \mathfrak{S}_\infty$, whose restriction to the diagonal subgroup is a direct sum of tensor representations of $\mathfrak{S}_\infty$; or perhaps it is better to say that those restrictions could be extended to the group of all (infinite) permutations $\mathfrak{S}^\infty$. A typical example of such a representation is the irreducible right-left (regular) representation of $\mathfrak{S}_\infty \times \mathfrak{S}_\infty$ in $l^2(\mathfrak{S}_\infty)$, whose restrictions to each multiplier are type II_1 factor representations. Olshansky gave a method of description, but the whole list of admissible representations and a model for it was obtained by Okounkov. The answer is very natural and related to actions of the bisymmetric group in a groupoid. This was one of the first examples of a non-locally compact ("big") group of type 1 with a very rich set (of infinite dimension) of irreducible representations. By the way, the identification of the left and right parts of some of those representations with von Neumann factors is still an open question.

The description of the characters in Thoma's theorem mentioned above is equivalent to the description of so-called central measures of the space of Young tableaux (= space of paths of Young graphs)—or to the description of homomorphisms of the ring of symmetric functions Λ to the scalars that are positive on the Schur functions $s_\mu(.) \in \Lambda$. Each deformation of a Young graph (or one can substitute other symmetric functions for the Schur polynomials) gives a new problem

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of this type. With S. Kerov and G. Olshansky, Okounkov gave the solution of such a problem for Jack polynomials $P_\mu(x,\theta)\theta \geq 0$; in this case the formula above is changed as follows. For the supernewtonian sums $s_n(\alpha,\beta)$ one must substitute the sums $s_{n,\theta} = \sum_{k=1}^{\infty}[\alpha_k^n + (-\theta)^{n-1}\beta_k^n]$ where α and β have the same meaning and where the parameter $\theta \in [0,\infty)$. Similar problems about central measures on the space of paths of the graded graphs (Bratelli diagram)—in another words, the question of how to describe the traces on locally semisimple algebras—are difficult. Such problems were extensively studied by S. Kerov, G. Olshansky, A. Okounkov, the author, and others, but are not solved for many natural examples. They are very important for the asymptotic representation theory of classical groups and for the theory of symmetric functions. It is worth mentioning that the asymptotic point of view on the infinite symmetric group (also on the other groups of that type), produced a new approach to the theory of representations of the finite symmetric groups—in some sense this is a look from infinity to finiteness. This approach was started in the paper of Okounkov and the author (1996) and gave later many natural explanations of the classical results.

Several Formulas

Okounkov is the author (or coauthor) of papers containing remarkable concrete formulas that concern very classical subjects. Among other results by him about symmetric functions and their application to various problems of analysis and representations, I want to mention two examples of important formulas. In 1996-7 with G. Olshansky (following Olshansky-Kerov and other authors), Okounkov systematically studied the theory of shifted Schur functions and its role in representation theory and obtained an elegant new formula for the number of standard tableaux of a skew Young diagram $\lambda \vdash n, \mu \vdash k, \lambda \supset \mu$:

$$\dim \lambda/\mu = \frac{s_\mu^*(\lambda)}{n(n-1)\dots(n-k+1)}.$$

This formula is mainly based on the surprising vanishing theorem of Okounkov:

$$s_\mu^*(\lambda) = 0 \quad \text{unless} \quad \mu \subset \lambda \quad s_\mu^*(\mu) = \prod_{\alpha\in\mu} h(\alpha).$$

Here $h(\alpha)$ is the hook length of the cell α of the diagram μ. The reformulation of that theory of shifted Schur functions in terms of Frobenius coordinates was given later (by Olshansky-Regev-Vershik) and led to the so-called inhomogeneous Frobenius-Schur symmetric functions. These are nothing but reformulations of the shifted Schur functions in Frobenius coordinates.

Another example is a formula that attracted the attention of many mathematicians because of its importance. This formula came out of Okounkov's and A. Borodin's beautiful answer to the question posed by Deift-Its about the existence of a formula for the determinant of a Toeplitz matrix as a Fredholm determinant. Borodin and Okounkov gave such a formula with an elegant proof. It happened that it had been proved before (with a different proof) by Jeronimo-Case, but this does not reduce the importance of their result.

Asymptotic Statistics of Young Diagrams with Respect to Plancherel Measure

This is one of the most impressive developments during the last few years in analysis and representation theory.

The study of the statistics of Young diagrams with Plancherel measure and the limit shape of typical diagrams was started in the papers of the author and S. Kerov in the 1970s. At the end of the 1990s, the remarkable link between those questions and the behavior of the eigenvalues of Gaussian random matrices was discovered. Great progress has been made in the last ten years, and the area has simultaneously been connected to random matrices, matrix problems, orthogonal polynomials, Young diagrams, integrable systems, determinant point processes, etc.

Many mathematicians took part in this progress: J. Baik, P. Deift, K. Johansson, C. Tracy, H. Widom, and others. Using serious analytical tools, those authors obtained striking and deep results about distributions of the fluctuations of the maximal eigenvalue of Gaussian matrices (Tracy-Widom) and the first and second row of a Young diagram with respect to Plancherel measure (Baik-Deift-Johansson). The contribution of Okounkov to subsequent progress was very important: *he suggested a strategy for considering all such problems*. This strategy, together with previous ideas of his coauthors G. Olshansky and A. Borodin, led to completely new proofs and simplifications of previous results on the one hand, and on the other hand provided a jumping-off point for him to move from those problems to random surfaces, to algebraic curves, to the Gromov-Witten/Hurwitz correspondence, etc. He was able to join together problems that had previously been very far from each other. One can say that this was a continuation of the "representation-theoretic" and the "partition-combinatorial" approach to the problems, together with ideas from modern mathematical physics.

I will mention only the main result. The concrete problem was to refine old results about the limit distribution of the spectrum of random Gaussian matrices (Wigner's semicircle law) and the limit shape of random Young diagrams with Plancherel measure (Vershik-Kerov, Logan-Shepp), and to find the distribution of the fluctuations of the maximal eigenvalue of random matrices (or the first several rows of Young diagrams). In a few preliminary papers Okounkov prepared the exclusively natural approach to the precise calculation of the correlation functions.

The correlation functions for random point process had been studied earlier for so-called z-measures on Young diagrams, in several papers of Borodin-Olshansky. The idea was to extend their calculation to Plancherel measure. Consider "poissonization" (=passage to a grand canonical ensemble) of the Plancherel measure on the diagrams. The so-called "Russian way" to look at a Young diagram is to turn the diagram 135°, and then to project it onto the lattice $\mathbb{Z}$, or, more exactly, onto the lattice $\mathbb{Z} + (1/2)$. When diagrams are equipped with poissonization of the Plancherel measure, the "Russian way" gives a remarkable random point process on the lattice, which is a determinant process (even before the limit procedure!). This is the main point, and it gives the possibility of calculating correlation functions of the processes and of obtaining a remarkable solution of the problem about the fluctuation of the Plancherel Young diagrams in the middle of the diagrams and near the edges. It gives correspondingly the Airy and Bessel ensemble as a determinant process on the lattice $\mathbb{Z} + (1/2)$ and produces as a partial result the proof of Deift's conjecture about joint distributions of the fluctuations of finitely many of the rows of a Young diagram with respect to Plancherel measure. This technique avoids the complicated tools of the Riemann-Hilbert problem as well as other tools that were used before, and brings to light the essence of the effects. Okounkov's idea about a direct link with random matrices was realized via matrix problems and ramified covering of surfaces. We can say now that the asymptotic theory of the Plancherel measure on Young diagrams is more or less complete.

In order to give a flavor of the ideas in those papers of Okounkov I mention some links and preliminary and hidden ideas: the action of $SL(2)$ on the partitions (started by S. Kerov), the infinite wedge model, boson-fermion correspondence; the very important new idea of Schur measures on the diagrams and, later, its generalization to Schur processes, which allowed one to consider three-dimensional Young diagrams and their limit shapes, asymptotics of enumeration of the ramified coverings of surfaces, etc. Of course this gigantic body of work still is not in complete order, and many ideas must be made clearer. But the number of results and future prospects are impressive.

The Work of Grigory Perelman

*John W. Morgan**

Introduction

I will report on the work of Grigory Perelman for which he was awarded the Fields Medal at the International Congress in the summer of 2006. Perelman posted three preprints on the arXiv between November 2002 and July 2003, [14, 16, 15]. In these preprints he gave a complete, albeit highly condensed, proof of the Poincaré Conjecture. Furthermore, at the end of the second preprint he stated a theorem about three-manifolds with curvature bounded below which are sufficiently collapsed. He showed how, from what he had established in the first two of his preprints, this collapsing result would imply the vast generalization of the Poincaré Conjecture, known as Thurston's Geometrization Conjecture. He stated that he would provide another manuscript proving the collapsing result, and in private conversations, he indicated that the proof used ideas contained in an earlier unpublished manuscript of his from 1992 and a then recently circulated manuscript of Shioya-Yamaguchi (which has now appeared [20]). To date, Perelman has not posted the follow-up paper establishing the collapsing result he stated at the end of his second preprint.

In this article, I will explain a little of the history and significance of the Poincaré Conjecture and Thurston's Geometrization Conjecture. Then I will describe briefly the methods Perelman used to establish these results, and I will discuss my view of the current state of the Geometrization Conjecture. Lastly, I will speculate on future directions that may arise out of Perelman's work. For a survey on the Poincaré Conjecture see [12]. For more details of the ideas and results we sketch here, the reader can consult [10], [2], and [13] and [23].

The Poincaré Conjecture and Thurston's Geometrization Conjecture

In 1904 Poincaré asked whether every closed (i.e., compact without boundary) simply connected three-manifold is homeomorphic to the three-sphere, see [18]. (It has long been known that this is equivalent to asking whether a simply connected smooth three-manifold is diffeomorphic to the three-sphere.) What has come to be known as the Poincaré Conjecture is the conjecture that the answer to this question is "yes". Since its posing, the Poincaré Conjecture has been a central problem in topology, and most of the advances in study of the topology of manifolds, both in dimensions three and in higher dimensions, over the last one hundred

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years have been related to the Poincaré Conjecture or its various generalizations. Prior to Perelman's work, analogues of the conjecture had been formulated in all dimensions and the topological versions of these analogues had been established (see [21] and [6]) in all dimensions except dimension three. Counterexamples to the analogue of the Poincaré Conjecture for smooth manifolds of higher dimension were first given by Milnor, in [11], where he constructed exotic smooth structures on the 7-dimensional sphere. The smooth four-dimensional Poincaré Conjecture remains open. Perelman's work resolves one remaining case of the topological version of the question. The Poincaré Conjecture is also the first of the seven Clay Millennium problems to be solved.

Poincaré was motivated to ask his question by an attempt to characterize the simplest of all three-manifolds, the three-sphere. But there is no reason to restrict only to this three-manifold. In 1982 Thurston formulated a general conjecture, known as Thurston's Geometrization Conjecture, which, if true, would (essentially) classify all closed three-manifolds, see [24] and [19]. Unlike the Poincaré Conjecture, which has a purely topological conclusion, Thurston's Conjecture has a geometric conclusion: In brief, it says that every closed three-manifold has an essentially unique two-step decomposition into simpler pieces. The first step is to cut the manifold open along a certain family of embedded two-spheres and cap off the resulting boundaries with three-balls. The second step is to cut the manifold open along a certain family of two-tori. The conjecture posits that at the end of this process each of the resulting pieces will admit a complete, finite-volume, Riemannian metric locally modeled on one of the eight homogeneous three-dimensional geometries. All manifolds modeled on seven of the eight geometries are completely understood and easily listed. The eighth homogeneous three-dimensional geometry is hyperbolic geometry. It is the richest class of examples and is the most interesting case. Manifolds modeled on this are given as quotients of hyperbolic three-space by torsion-free lattices in $SL(2, \mathbf{C})$ of finite co-volume. The classification of these manifolds is equivalent to the classification of such lattices, a problem that has not yet been completely solved. Thurston's Conjecture includes the Poincaré Conjecture as a special case, since the only geometry that can model a simply connected manifold is the spherical geometry (constant positive curvature), and simply connected manifolds of constant positive curvature are easily shown to be isometric to the three-sphere.

Over the last one hundred years there have been many attempts to prove the Poincaré Conjecture. Most have been direct topological attacks which involve simplifying surfaces and/or loops in a simply connected three-manifold in order to show that the manifold is the union of two three-balls, which then implies that it is homeomorphic to the three-sphere. While these types of topological arguments have proved many beautiful results about three-dimensional manifolds, including for example that a knot in the three-sphere is trivial (i.e., unknotted) if and only if the fundamental group of its complement is isomorphic to $\mathbb{Z}$, they have said nothing about the Poincaré Conjecture. For a description of some of these approaches and why they failed see [22]. More recently, there have been approaches to the Geometrization Conjecture along the following lines. Given any three-manifold, the complement of a sufficiently general knot in the manifold will have a complete hyperbolic structure of finite volume. Work from here back to the original manifold by navigating in the space of hyperbolic structures with cone-like singularities. This approach, pioneered by Thurston, proved a beautiful result, called the orbifold theorem [5], that proves a special case of the Geometrization Conjecture, but has not led to a proof of the full conjecture.

Perelman's Method: Ricci Flow

How did Perelman do it? He used, and generalized, the work of Richard Hamilton on Ricci flow. For an introduction to Ricci flow see [3] and for a collection of Hamilton's papers on Ricci flow see [1]. Hamilton showed, at least in good cases, that the Ricci flow gives an evolution of a Riemannian metric on a 3-manifold that converges to a locally homogeneous metric (e.g., a metric of constant positive curvature), see [7] and [9]. But Hamilton also showed (see [8]) that there are other possibilities for the effect of Ricci flow. It can happen, and does happen, that the Ricci flow develops finite-time singularities. These finite-time singularities impede the search for a good metric; for, in order to find the good metric one needs to continue the flow for all positive time—the Riemannian metrics one is looking for in general only appear as limits as time goes to infinity. In fact, the finite-time singularities are needed in order to do the first layer of cutting (along two-spheres and filling in balls) as required in Thurston's Conjecture. These cuttings along two-spheres and filling in balls have been long known to be necessary in order to produce a manifold that admits a Riemannian metric modeled on one of the homogeneous geometries. As we shall see presently, the second step in the decomposition proposed by Thurston, along two-tori, happens as t limits to infinity.

The Ricci flow equation, as introduced and first studied by Hamilton in [7], is a weakly parabolic partial differential equation for the evolution of a Riemannian metric g on a smooth manifold. It is given by:

$$\frac{\partial g(t)}{\partial t} = -2\mathrm{Ric}(g(t)),$$

where $\mathrm{Ric}(g(t))$ is the Ricci curvature of $g(t)$. One should view this equation as a (weakly) nonlinear version of the heat equation for symmetric two-tensors on a manifold. Hamilton laid down the basics of the theory of this equation—short-time existence and uniqueness of solutions. Furthermore, using a version of the maximum principle he established various important differential inequalities, including a Harnack-type inequality, that are crucial in Perelman's work. Hamilton also understood that, in dimension three, surgery would be a crucial ingredient for two reasons: (i) it is the way to deal with the finite-time singularities and (ii) because of the necessity in general to do cutting on a given three-manifold in order to find the good metric. With this motivation, Hamilton introduced the geometric surgeries that Perelman employs.

From the point of view of the previous paragraph, an n-dimensional Ricci flow is a one-parameter family of metrics on a smooth manifold. It is also fruitful to take the perspective of space-time and define an n-dimensional Ricci flow as a horizontal metric on $M \times [a, b]$ where the metric on $M \times \{t\}$ is the metric $g(t)$ on M. This point of view allows for a natural generalization: a generalized n-dimensional Ricci flow is an $(n+1)$-dimensional space-time equipped with a time function which is a submersion onto an interval, a vector field which will allow us to differentiate in the "time direction", and a metric on the "horizontal distribution" (which is the kernel of the differential of the time function). We require that these data be locally isomorphic to a Ricci flow on a product $M \times (a, b)$, with the vector field becoming $\partial/\partial t$ in the local product structure.

Perelman's First Two Preprints

Here we give more details on Perelman's first two preprints, [14] and [16]. As we indicated above, the work in these preprints is built on the foundation of Ricci flow as developed by Hamilton. A central concept is the notion of an n-dimensional Ricci flow, or generalized Ricci flow, being κ-non-collapsed on scales $\leq r_0$ for some $\kappa > 0$. Let p be a point of space-time and let t be the time of p. Denote by $B(p, t, r)$ the metric ball of radius r in the t time-slice of space-time, and denote by $P(x, t, r, -r^2)$ the backwards parabolic neighborhood consisting of all flow lines backwards in time from time t to time $t - r^2$ starting at points of $B(x, t, r)$. This means that for any point p and any $r \leq r_0$ the following holds. If the norm of the Riemannian curvature tensor on $P(p, t, r, -r^2)$ is at most r^{-2} then the volume of $B(x, t, r)$ is at least κr^n. Here are the main new contributions of his first two preprints:

(1) He introduced an integral functional, called the reduced $\mathcal{L}$-length for paths in the space-time of a Ricci flow.

(2) Using (1) he proved that for every finite time interval I, a Ricci flow of compact manifolds parameterized by I are non-collapsed where both the measure of non-collapsing and the scale depend only on the interval and the geometry of the initial manifold.

(3) Using (1) and results of Hamilton's on singularity development, he classified, at least qualitatively, all models for singularity development for Ricci flows of compact three-manifolds at finite times. These models are the three-dimensional, ancient solutions (i.e., defined for $-\infty < t \leq 0$) of non-negative, bounded curvature that are non-collapsed on all scales ($r_0 = \infty$).

(4) He showed that any sequence of points (x_n, t_n) in the space-time of a three-dimensional Ricci flow whose times t_n are uniformly bounded above and such that the norms of the Riemannian curvature tensors $Rm(x_n, t_n)$ go to infinity (a so-called "finite-time blow-up sequence") has a subsequence converging geometrically to one of the models from (3). This result gives neighborhoods, called "canonical neighborhoods", for points of sufficiently high scalar curvature. These neighborhoods are geometrically close to corresponding neighborhoods in non-collapsed ancient solutions as in (3). Their existence is a crucial ingredient necessary to establish the analytic and geometric results required to carry out and to repeat surgery.

(5) From the classification in (3) and the blow-up result in (4), he showed that surgery, as envisioned by Hamilton, was always possible for Ricci flows starting with a compact three-manifold.

(6) He extended all the previous results from the category of Ricci flows to a category of certain well-controlled Ricci flows with surgery.

(7) He showed that, starting with any compact three-manifold, repeatedly doing surgery and restarting the Ricci flow leads to a well-controlled Ricci flow with surgery defined for all positive time.

(8) He studied the geometric properties of the limits as t goes to infinity of the Ricci flows with surgery.

Let us examine these in more detail. Perelman's length function is a striking new idea, one that he has shown to be extremely powerful, for example, allowing him to prove non-collapsing results. Let $(M, g(t))$, $a \leq t \leq T$, be a Ricci flow. The reduced $\mathcal{L}$-length functional is defined on paths $\gamma\colon [0, \bar{\tau}] \to M \times [a, T]$ that are parameterized by backwards time, namely satisfying

$\gamma(\tau) \in M \times \{T - \tau\}$ for all $\tau \in [0, \bar{\tau}]$. One considers

$$l(\gamma) = \frac{1}{2\sqrt{\bar{\tau}}}\int_0^{\bar{\tau}} \sqrt{\tau}\,(R(\gamma(\tau)) + |X_\gamma(\tau)|^2)\,d\tau.$$

Here, $R(\gamma(\tau))$ is the scalar curvature of the metric $g(T - \tau)$ at the point in M that is the image under the projection into M of $\gamma(\tau)$, and $X_\gamma(\tau)$ is the projection into TM of the tangent vector to the path γ at $\gamma(\tau)$. The norm of $X_\gamma(\tau)$ is measured using $g(T - \tau)$. It is fruitful to view this functional as the analogue of the energy functional for paths in a Riemannian manifold. Indeed, there are $\mathcal{L}$-geodesics, $\mathcal{L}$-Jacobi fields, and a function $l_{(x,T)}(q,t)$ that measures the minimal reduced $\mathcal{L}$-length of any $\mathcal{L}$-geodesic from (x,T) to (q,t). Fixing (x,T), the function $l_{(x,T)}(q,t)$ is a Lipschitz function of (q,t) and is smooth almost everywhere.

Perelman proves an extremely important monotonicity result along $\mathcal{L}$-geodesics for a function related to this reduced length functional. Namely, suppose that W is an open subset in the time-slice $T - \bar{\tau}$, and each point of W is the endpoint of a unique minimizing $\mathcal{L}$-geodesic. Then he defined the reduced volume of W to be

$$\int_W (\bar{\tau})^{-n/2} e^{-l_{(x,T)}(w,T-\bar{\tau})}\,dvol.$$

For each $\tau \in (0, \bar{\tau}]$ let $W(\tau)$ be the result of flowing W along the unique minimal geodesics to time $T - \tau$. Then the reduced volume of $W(\tau)$ is a monotone non-increasing function of τ. This is the basis of his proof of the non-collapsing result: Fix a point (x,T) satisfying the hypothesis of non-collapsing for some $r \le r_0$. The monotonicity allows him to transfer lower bounds on reduced volume (from (x,T)) at times near the initial time (which is automatic from compactness) to lower bounds on the reduced volume times near T. From this, one proves the non-collapsing result at (x,T).

Perelman then turned to the classification of ancient three-dimensional solutions (ancient in the sense of being defined for all time $-\infty < t \le 0$) of bounded, non-negative curvature that are non-collapsed on all scales. By geometric arguments he established that the space of based solutions of this type is compact, up to rescaling. Here the non-collapsed condition is crucial: After we rescale to make the scalar curvature one at the base point, this condition implies that the injectivity radius at the base point is bounded away from zero. There are several types of these ancient solutions. A fixed time section is of one of the following types—(i) a compact round three-sphere or a Riemannian manifold finitely covered by a round three-sphere, (ii) a cylinder which is a product of a round two-sphere with the line or a Riemannian manifold finitely covered by this product, (iii) a compact manifold diffeomorphic to S^3 or $\mathbb{R}P^3$ that contains a long neck which is approximately cylindrical, (iv) a compact manifold of bounded diameter and volume (when they are rescaled to have scalar curvature 1 at some point), and (v) a non-compact manifold of positive curvature which is a union of a cap of bounded geometry (modulo rescaling) and a cylindrical neck (approximately S^2 times a line) at infinity.

Using the non-collapsing result and delicate geometric limit arguments, Perelman showed that in a Ricci flow on compact three-manifolds any finite-time blow-up sequence has a subsequence which, after rescaling to make the scalar curvature equal to one at the base point, converges geometrically to a non-collapsed, ancient solution of bounded, non-negative curvature. (Geometric convergence means the following: Given a finite-time blow-up sequence (x_n, t_n), after replacing the sequence by a subsequence, there is a model solution with base point and scalar curvature at the base point equal to one such that for any $R < \infty$ the balls of radius R centered at the x_n in the n^{th} rescaled flow converge smoothly to a ball of radius R centered at the base point in the given model.) For example, we could have a component of positive Ricci curvature. According to Hamilton's result [7] this manifold contracts to a point at the singular time and as it does so it approaches a round (i.e., constant positive curvature) metric. Thus, the model in this case is a round manifold with the same fundamental group. Perelman's result implies that there is a threshold so that all points of scalar curvature above the threshold have canonical neighborhoods modeled on corresponding neighborhoods in non-collapsed ancient solutions. This leads to geometric and analytic control in these neighborhoods, which in turn is crucial for the later arguments showing that surgery is always possible.

Now let us turn to surgery. An n-dimensional Ricci flow with surgery is a more general object. It consists of a space-time which has a time function. The level sets of the time function are called the time-slices and they are compact n-manifolds. This space-time contains an open dense subset that is a smooth manifold of dimension $n+1$ equipped with a vector field and a horizontal metric that make this open dense set a generalized n-dimensional Ricci flow. At the singular times, the time-slices have points not included in the open subset which is a generalized Ricci flow. This allows the topology of the time-slices to change as the time evolves past a singular time. For example, in the case just described above, when a component is shrinking to a point, the effect of surgery is to remove entirely that component. A more delicate case is when the singularity is modeled on a manifold with a long, almost cylindrical tube in it. In this case, at the singular time, the singularities are developing inside the tube. Following Hamilton, at the singular time Perelman cuts off the tube near its ends and sews in a predetermined metric on the three-ball, using a partition of unity. The effect of surgery is to produce a new, usually topologically distinct manifold at the singular, or surgery, time. After having produced this new closed, smooth manifold, one restarts the Ricci flow using that manifold as the initial conditions. This then is the surgery process: remove some components of positive curvature and those fibered over circles by manifolds of positive curvature (all of which are topologically standard) and surger others along two-spheres, and then restart Ricci flow.

Perelman then showed the entire Ricci flow analysis described above extends to Ricci flows with surgery, provided that the surgery is done in a sufficiently controlled manner. Thus, one is able to repeat the argument ad infinitum and construct a Ricci flow with surgery defined for all time. Furthermore, one has fairly good control on both the change in the topology and the change in the geometry as one passes the surgery times. Here then is the main result of the first two preprints taken together:

Theorem. *Let (M, g_0) be a compact, orientable Riemannian three-manifold. Then there is a Ricci flow with surgery $(\mathcal{M}, G)$ defined for all positive time whose zero-time slice is (M, g_0). This Ricci flow with surgery has only finitely many surgery times in any compact interval. As one passes a surgery time, the topology of the time-slices changes in the following manner. One does a finite number of surgeries along disjointly embedded two-spheres (removing an open collar neighborhood of the two-spheres and gluing three-balls along each of the resulting boundary two-spheres) and removes a finite number of topologically standard components (i.e., components admitting round metrics and components finitely covered by $S^2 \times S^1$).*

A couple of remarks are in order:

(i) It is clear from the construction that if Thurston's Geometrization Conjecture holds for the manifold at time t then it holds for the manifolds at all previous times.

(ii) In this theorem one does not need orientability, only the weaker condition that every projective plane has non-trivial normal bundle. To extend the results to cover manifolds admitting projective planes with trivial normal bundle one takes a double covering and works equivariantly. This fits perfectly with the formulation of the Geometrization Conjecture for such manifolds.

In addition, Perelman established strong geometric control over the nature of the metrics on the time-slices as time tends to infinity. In particular, at the end of the second preprint he showed that for t sufficiently large, the t time-slice contains a finite number of incompressible tori (incompressible means π_1 injective) that divide the manifold into pieces. Each component that results from the cutting process has a metric of one of two types. Either the metric is, after rescaling, converging to a complete constant negatively curved metric, or the metric is arbitrarily collapsed on the scale of its curvature. Components of the first type clearly support hyperbolic metrics of finite volume. It is to deal with the components of the second type that Perelman states the proposed result on collapsed manifolds with curvature bounded below.

Completion of the Proof of the Poincaré Conjecture

As we have already remarked, the Poincaré Conjecture follows as a special case of Thurston's Geometrization Conjecture. Thus, the results of the first two of Perelman's preprints together with the collapsing result stated at the end of the second preprint, give a proof of the Poincaré Conjecture. In a third preprint [15] Perelman gave a different argument, avoiding the collapsing result, proving the Poincaré Conjecture but not the entire Geometrization Conjecture. For a detailed proof along these lines, see Chapter 18 of [13]. One shows that if one begins with a homotopy three-sphere, or indeed any manifold whose fundamental group is a free product of finite groups and infinite cyclic groups, then the Ricci flow with surgery, which by the results of Perelman's first two preprints is defined for all positive time, becomes extinct after a finite time. That is to say, for all t sufficiently large, the manifold at time t is empty. We conclude that the original manifold is a connected sum of spherical space-forms (quotients of the round S^3 by finite groups of isometries acting freely) and S^2-sphere bundles over S^1. It follows that if the original manifold is simply connected, then it is diffeomorphic to a connected sum of three-spheres, and hence is itself diffeomorphic to the three-sphere. This shows that the Poincaré Conjecture follows from this finite-time extinction result together with the existence for all time of a Ricci flow with surgery.

Let us sketch how the finite-time extinction result is established in [15]. (There is a parallel approach in [4] using areas of harmonic two-spheres of non-minimal type instead of area minimizing 2-disks.) We consider the case of a homotopy three-sphere M (though the same ideas easily generalize to cover all cases stated above). The fact that the manifold is a homotopy three-sphere implies that $\pi_3(M)$ is non-trivial. Perelman's argument is to consider a non-trivial element in $\xi \in \pi_3(M)$. Represent this element by a two-sphere family of homotopically trivial loops. Then for each loop take the infimum of the areas of spanning disks for the loop, maximize over the loops in family and then minimize over all representative families for ξ. The result is an invariant $W(\xi)$. One asks what happens to this invariant under Ricci flow. The result (following similar arguments that go back to Hamilton [9]) is that

$$\frac{dW(\xi)}{dt} \leq -2\pi - \frac{1}{2}R_{\min}(t)W(\xi).$$

Here $R_{\min}(t)$ is the minimum of the scalar curvature at time t. Since one of Hamilton's results using the maximum principle is that $R_{\min}(t) \geq -6/(4t+\alpha)$ for some positive constant α, it is easy to see that any function satisfying Equation (6) goes negative in finite time. Perelman shows that the same equation holds in Ricci flows with surgery as long as the manifold continues to exist in the Ricci flow with surgery. On the other hand, $W(\xi)$ is always non-negative. It follows that the homotopy three-sphere must disappear in finite time.

This then completes Perelman's proof of the Poincaré Conjecture. Start with a homotopy three-sphere. Run the Ricci flow with surgery until the manifold disappears. Hence, it is a connected sum of manifolds admitting constant positive curvature metrics. That is to say it is a connected sum of three-spheres, and hence itself diffeomorphic to the three-sphere.

Status of the Geometrization Conjecture

What about the general geometrization theorem? Perelman's preprints show that proving this result has been reduced to proving a statement about manifolds with curvature locally bounded below that are collapsed in the sense that they have short loops through every point. There are results along these lines by Shioya-Yamaguchi [20]. The full result that Perelman needs can be found in an appendix to [20], except for the issue of allowing a boundary. The theorem that Perelman states in his second preprint allows for boundary tori, whereas the statement in [20] is for closed manifolds. Perelman has indicated privately that the arguments easily extend to cover this more general case. But in any event, invoking deep results from three-manifold topology, it suffices to consider only the closed case in order to derive the full Geometrization Conjecture. The paper [20] relies on an earlier, unpublished work by Perelman.

All in all, Perelman's collapsing result, in the full generality that he stated it, seems eminently plausible. Still, to my mind the entire collapsing space theory has not yet received the same careful scrutiny that Perelman's preprints have received. So, while I see no serious issues looming, I personally am not ready to say that geometrization has been completely checked in detail. I am confident that it is only a matter of time before these issues are satisfactorily explored.

The Effect of Perelman's Work

Let me say a few words about the larger significance of what Perelman has accomplished and what the future may hold. First of all, an affirmative resolution of the one-hundred-year-old Poincaré Conjecture is an accomplishment rarely equaled or surpassed in mathematics. An affirmative resolution of the Geometrization Conjecture will surely lead to a complete and reasonably effective classification of all closed three-dimensional manifolds. It is hard to overestimate the progress that this represents. The Geometrization Conjecture is a goal in its own right, leading as it (essentially) does to a classification of three-manifolds. Paradoxically, the effect of Perelman's work on three-manifold topology will be minimal. Almost all workers in the field were already assuming that the Geometrization Conjecture is true and were working modulo that assumption, or else they were working directly on hyperbolic three-manifolds, which obviously satisfy the Geometrization Conjecture.

The largest effects of Perelman's work will lie in other applications of his results and methods. I think there are possibilities for applying Ricci flow to four-manifolds. Four-manifolds are terra incognita compared to three-manifolds. In dimension four there is not even a guess as to what the possibilities are. Much less is known, and what is known suggests a far more complicated landscape in dimension four than in dimension three. In order to apply Ricci flow and Perelman's techniques to four-manifolds there are many hurdles to overcome. Nevertheless, this is an area that, in my view, shows promise.

There are also applications of Ricci flow to Kähler manifolds, where already Perelman's results are having an effect—see for example [17] and the references therein.

As the above description should make clear, Perelman's great advance has been in finding a way to control and qualitatively classify the singularities that develop in the Ricci flow evolution equation. There are many evolution equations in mathematics and in the study of physical phenomena that are of the same general nature as the Ricci flow equation. Some, such as the mean curvature flow, are related quite closely to the Ricci flow. Singularity development is an important aspect of the study of almost all of these equations both in mathematics and in physical applications. It is not yet understood if the type of analysis that Perelman carried out for the Ricci flow has analogues in some of these other contexts, but if there are analogues, the effect of these on the study of those equations could be quite remarkable.

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The Work of Terence Tao

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The work of Terence Tao includes major contributions to analysis, number theory, and aspects of representation theory. Perhaps his most spectacular achievement to date is the proof that the set of prime numbers contains arbitrarily long arithmetic progressions (jointly with Ben Green). A reasonably detailed account on this result would already easily make up the full article. But it certainly would not give a picture of the unique scope and diversity in Tao's opus. The number of areas that he has marked either by solving the main questions or making them progress in a decisive way is utterly astonishing. This unique problem-solving ability relies not only on supreme technical strength but also on a deep global understanding of large parts of mathematics. To illustrate this, I plan to report below also on some of his contributions to harmonic analysis, partial differential equations, and representation theory. Because of lack of space, the results will be formulated with little or no background discussion or how they fit in the larger picture of Tao's research.

1) Tao has the strongest results to date on several central conjectures in higher-dimensional Fourier analysis, first formulated by E. Stein. These conjectures express mapping properties of the Fourier transform when restricted to hypersurfaces (in particular, the Fourier restriction conjecture to spheres and the Bochner-Riesz conjectures belong to this class of problems). Those issues were basically understood in dimension 2 already in the 1970s, but they turn out to be much more resistant starting from dimension 3 (see [1] for the current state of affairs). They are intricately connected to combinatorial questions such as the dimension of higher-dimensional "Kakeya sets". Those are compact subsets in R^n containing a line segment in every direction. Such sets may have zero Lebesgue measure (the well-known Besicovitch construction in the plane) but it is believed that the Hausdorff dimension is always maximal (i.e., equals n). This

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problem, which underlies Stein's conjectures, has been the object of intensive research over the last few decades. Although still unsolved even for $n = 3$, also here Tao's work has led to some of the most significant developments, often of interest in their own right. Among these is his joint paper with N. Katz and I. Laba [2] and the discovery of the "sum-product phenomenon" in finite fields [3].

2) Tao's work on nonlinear dispersive equations is too extensive to describe here in any detail. But the two papers [4], [5] particularly stand out. Both prove a final result, contain new ideas, and are technically speaking a real tour de force. The paper [4] (jointly with J. Colliander, M. Keel, G. Staffilani and H. Takaoka) solves the longstanding conjecture of global wellposedness and scattering for the defocusing energy-critical Schrödinger equation in R^3, thus the equation $iu_t + \Delta u - u|u|^4 = 0$. This is the counterpart for the nonlinear Schrödinger equation of M. Grillakis' theorem [6] on nonlinear wave equations. In both cases, the local theory was understood for at least a decade, and the main difficulty was to rule out "energy concentrations". Tao and his collaborators resolve this issue by proving a new type of "Morawetz-inequality" where the solution u is studied simultaneously in physical and frequency space. In the paper [5] the wave map equation with spherical target is studied in dimension 2, in which case the energy norm is the critical Sobolev norm for the local wellposedness of the Cauchy problem. This result is the main theorem of the paper. Its proof (which builds upon several earlier works) involves a careful study of the "null-structures" in the nonlinearity and a novel renormalization argument. One should mention that large data may create blowup behavior although this is not expected to happen if the sphere is replaced by a hyperbolic target (S. Klainerman's conjecture).

3) Tao's proof (in joint work with B. Green) that the prime numbers contain arbitrarily long arithmetic progressions (a problem considered out of reach even for length-four progressions) has come as a real surprise to the experts, because the initial breakthrough here is one in ergodic theory rather than in our analytical understanding of prime numbers. Of great relevance is Szemerédi's theorem, which roughly states that arbitrary sets of integers of positive density contain arithmetic progressions of any length (as conjectured by Erdős and Turán). Szemerédi's original proof was combinatorial. Subsequent developments came with H. Furstenberg's new proof through ergodic theory and the concept of multiple recurrence and later T. Gowers' approach using harmonic analysis, closer in spirit to K. Roth's work. All of them are crucial for the discussion of the Green-Tao achievement. In their first paper [7], the existence of progressions in the primes is derived from a remarkable extension of Szemerédi's theorem, claiming that the same conclusion holds if now we consider subsets of positive density in a "pseudo-random" set. Here one should be more precise about what "pseudo-random" means, but let us just say that the prime numbers fit the model well, and the required pseudo-random "container" is simply provided by looking at an appropriate set of pseudo-primes (these are integers without small divisors). Following Furstenberg's structural approach (in a finite setting), Green and Tao construct in this pseudo-random setting a "characteristic factor" (in the sense of Furstenberg's theory) that is of positive density in the integers and hence allows application of Szemerédi's theorem. Their analysis uses strongly the so-called "Gowers norms" introduced in [8] to manipulate higher-order correlations. The importance of [8] is even more prominent in Green and Tao's subsequent papers (see [9] in particular) where [7] is combined with Gowers' analysis involving generalized spectra and classical techniques from analytic number theory. The authors establish in particular the validity of the expected asymptotic formula for the number of length 4 progressions in the primes. Such a formula had been obtained by Van der Corput in 1939 for prime triplets, using the circle method and Vinogradov's work. Besides solving questions of historical magnitude, the work of Green and Tao had a strong unifying effect on various fields.

4) I conclude this report with one more line of research in a completely different direction: Tao's solution [10] (jointly with A. Knutsen) of the "saturation conjecture" for the Littlewood-Richardson coefficients in representation theory, formulated by A. A. Klyachko in 1998, and Horn's conjecture for the eigenvalues of hermitian matrices under addition. This was one of Tao's earlier achievements (with a later follow-up). Again it has greatly impressed experts in the field.

So much in such short time. To paraphrase C. Fefferman, "What's next?"

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The Work of Wendelin Werner

*Harry Kesten**

Wendelin Werner received the Fields Medal for work which is on the borderline of probability and statistical physics, and which confirmed and made rigorous quite a number of conjectures made by mathematical physicists studying critical phenomena. In particular he proved so-called power laws and gave the precise value of corresponding critical exponents.

Many of Werner's papers are joint papers with one or more of G. Lawler, O. Schramm, and S. Smirnov. Together they have developed entirely new tools for determining limits of lattice models as the lattice spacing goes to 0, and for establishing conformal invariance of such limits. Even though there are still many open problems, the work of Werner and his co-workers opened up an area in which no rigorous arguments were known.

Werner received a Fields Medal for a theory and a general method, rather than for a specific theorem. We can only discuss some of this work; basically we restricted ourselves to Werner's work from the year 1999 on. More details, and especially references to other contributors than Werner and his co-workers can be found in [11], [12], and [3].

Background: Stochastic Loewner Evolutions or Schramm-Loewner Evolutions

Schramm [8] invented a new kind of stochastic process to model certain two-dimensional random processes, in which the state at time t is the initial piece of a curve $\gamma : [0,t] \to \mathbb{C}$, or more generally, a set $S(t) \subset \mathbb{C}$ which increases with t. A principal motivating example was the so-called scaling limit of loop-erased random walk on $\delta\mathbb{Z}^2$ as $\delta \downarrow 0$. Loop-erased random walk on $\delta\mathbb{Z}^2$ starting from $a \in \delta\mathbb{Z}^2$ and stopped at a set K is obtained as follows: First we choose a simple random walk path from a to K. Let $S_0 = a$. If $S_0, S_1, \dots, S_k \in \delta\mathbb{Z}^2 \setminus K$ have already been chosen, pick S_{k+1} uniformly from the four neighbors of S_k in $\delta\mathbb{Z}^2$. Stop this process at the random time τ when $S_.$ first reaches a point in K. Then $S_0, \dots, S_\tau$ is a realization of simple random walk from a to K. A realization of loop-erased random walk is obtained by erasing the loops in $S_0, \dots, S_\tau$ in the order in which they arise. A formal description of this process can be found in [11].

The resulting path will be a self-avoiding path $LE_0, LE_1, \dots, LE_\sigma$ from a to K with $LE_j = S_{t(j)}$ for some $t(0) = 0 < t(1) < \cdots < t(\sigma) = \tau$. The process of loop-erased walks obtained in this way is completely described by its distribution function which is some probability measure on self-avoiding paths from a to K, which we denote by $LERW(a,K)$. Usually one wants to study the path as long as it is inside some domain $D \subset \mathbb{C}$, so one often takes K to be $\mathbb{C} \setminus D$ or an approximation to ∂D. How does a "typical" loop-erased path look? Does there exist a limit in a suitable topology of $LERW(a,K)$ as $\delta \downarrow 0$ and is the limit simple to describe? Such a limit is usually called a "scaling limit". Note that $\delta \downarrow 0$ automatically means that one studies walks of more and more steps. It seems reasonable to expect that the lattice structure of $\delta\mathbb{Z}^2$ is no longer visible in the limit and that the limit is rotation invariant. In fact, it was believed, in the first place by physicists, that the scaling limit exists and is even "conformally invariant". As Schramm puts it "conformal invariance conjectures [were] 'floating in the air'." [8] gives the following precise formulation to the conjecture (we skip any discussion of the topology in which the scaling limit should be taken):

> *Let $D \subsetneq \mathbb{C}$ be a simply connected domain in $\mathbb{C}$, and let $a \in D$. Then the scaling limit of $LERW(a,\partial D)$ exists. Moreover, suppose that $f : D \to D'$ is a conformal homeomorphism onto a domain $D' \subset \mathbb{C}$. Then $f_*\mu_{a,D} = \mu_{f(a),D'}$, where $\mu_{a,D}$ is the scaling limit measure of $LERW(a,\partial D)$ and $\mu_{f(a),D'}$ is the scaling limit measure of $LERW(f(a),\partial D')$.*

Schramm in [8] could not prove this conjecture, but assuming the conjecture to be true he did determine what $\mu_{a,D}$ had to be. (The conjecture has been proven now in [7].) Let $D \subsetneq \mathbb{C}$ be a simply connected domain containing 0 and let $\mathbb{U}$ be the open unit disc. By the Riemann mapping theorem there exists a unique conformal homeomorphism $\psi = \psi_D$ which takes D onto $\mathbb{U}$ and is such that $\psi(0) = 0$ and $\psi'(0)$ is real and strictly positive (the prime denotes differentiation with respect to the complex variable z). Now let $\eta : [0,\infty] \to \overline{\mathbb{U}}$ and set $U_t = \mathbb{U} \setminus \eta[0,t]$ and $g_t = \psi_{U_t}$. Then one can parametrize the path η in such a way that $g_t'(0) = e^t$. One can show that $W(t) := \lim_{z\to\eta(t)} g_t(z)$ exists and lies in

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$\partial\mathbb{U}$, where in this limit z approaches $\eta(t)$ from inside U_t. Loewner's slit mapping theorem implies that in the above parametrization of η, $g_t(z)$ satisfies the differential equation

$$\frac{\partial}{\partial t}g_t(z) = -g_t(z)\frac{W(t)+g_t(z)}{W(t)-g_t(z)},$$

$$g_0(z) = z \text{ for all } z \in \mathbb{U}.$$

Here $W : [0,\infty) \to \partial\mathbb{U}$ is some continuous function which is called the *driving function* of the curve η. Properties of η are tied to properties of W. If η is a random curve, then W is also random. To get back to loop-erased random walk, Schramm first showed that $\mu_{0,\mathbb{U}}$ has to be concentrated on simple curves. Using a kind of Markov property of loop-erased random walks and assuming the conjecture in italics above, Schramm then shows that if η (or rather its reversal) is chosen according to $\mu_{0,\mathbb{U}}$, then $W(t)$ must be $\exp[iB(\kappa t)]$ for a Brownian motion B whose initial point is uniformly distributed on $\partial\mathbb{U}$ and $\kappa \geq 0$ some constant. In a way one can get back from the driving force to a curve η. An SLE_κ process is then the process of the $\eta[0,t]$ (or of the sets $\mathbb{U} \setminus \eta[0,t]$) when the driving function $W(t)$ equals $B(\kappa t)$. What we described here is so-called *radial SLE*. There is also a variant called *chordal SLE* which we shall not define here.

What value of κ should one take for the scaling limit of loop-erased random walk? Schramm compares the winding number of SLE with that of $LERW$ to conclude that one should take $\kappa = 2$.

The Work of Werner on Brownian Intersection Exponents

It has turned out that the representation of certain models by SLE_κ is very helpful. Various questions reduce to questions about hitting probabilities of sets by SLE or behavior of functionals of SLE, and in many instances these can be reformulated into questions about stochastic differential equations and stochastic integrals. One now clearly needs techniques to show in concrete cases that a model can be analyzed by relating it to SLE. A case where Werner and his co-workers have been quite successful is the calculation of Brownian intersection exponents.

Without using SLE Lawler and Werner had already made good progress on determining the asymptotic behavior of the "non-intersection probability"

(2)

$$f(p_1,p_2) := P\Big\{\bigcup_{i=1}^{p_1} B_i^{(1)}[0,t] \cap \bigcup_{j=1}^{p_2} B_j^{(2)}[0,t] = \varnothing\Big\},$$

where $B_i^{(\ell)}, 1 \leq i \leq p_\ell, \ell = 1,2$, are independent planar Brownian motions with $B_i^{(\ell)}(0) = a_\ell \in \mathbb{H} := \mathbb{R}\times(0,\infty)$ with $a_1 \neq a_2$. They also studied the analogous non-intersection probabilities $f(p_1,\dots,p_q)$ for q "packets" of Brownian motions. It was known that there is some exponent $\xi(p_1,\dots,p_q) > 0$ such that $f(p_1,\dots,p_q) \approx t^{-\xi(p_1,\dots,p_q)/2}$. ($a \approx b$ means that $[\log b]^{-1}\log a$ tends to 1 as an appropriate limit is taken; here it is the limit as $t \to \infty$.) Lawler and Werner also considered $\tilde{f}(p_1,\dots,p_q)$ which is defined by adding the condition that all $B_{i_\ell}^{(\ell)}[0,t] \subset \mathbb{H}$ in (2). This has similar exponents $\tilde{\xi}$ such that $\tilde{f}(p_1,\dots,p_q) \approx t^{-\tilde{\xi}(p_1,\dots,p_q)/2}$. Duplantier and Kwon predicted (on the basis of non-rigorous arguments from quantum field theory and quantum gravity) explicit values for many of the exponents. This model has conformal invariance built in, because if $B(\cdot)$ is a planar Brownian motion in a domain D and Φ is a conformal homeomorphism from D onto D', then $\Phi\circ B(\cdot)$ is a time change of a Brownian motion in D'. In [4], [5] the authors manage to compute the exponents $\tilde{\xi}$ and ξ by relating Brownian excursions to SLE_6. An important ingredient of their proof is the singling out of SLE_6 as the only SLE process with the "locality property". Roughly speaking, this property says that SLE_6 does not "feel the boundary" of its domain as long as it does not hit it.

Earlier Lawler had shown that the Hausdorff dimension of various sets defined in terms of planar Brownian motions could be expressed by means of the exponents $\xi, \tilde{\xi}$ or similar exponents. For instance, the Hausdorff dimension of the outer boundary of a planar Brownian motion equals $2-\eta_2$. Here η_2 is the "disconnection" exponent defined by

$$P\{B^{(1)}[0,\sigma_R^{(1)}] \cup B^{(2)}[0,\sigma_R^{(2)}]$$
$$\text{does not disconnect 0 from } \infty\} \approx R^{-\eta_2},$$

with $B^{(1)}(0), B^{(2)}(0)$ two independent two-dimensional Brownian motions starting at 1, and $\sigma_R^{(i)}$ is the first hitting time by $B^{(i)}$ of the circle of radius R and center at the origin.

In [4], [5] it is shown that η_2 equals $2/3$, thereby proving a conjecture by Mandelbrot of more than twenty years' standing.

Background: Conformal Invariance in Percolation

Percolation is one of the simplest models which exhibits a phase transition and because of its simplicity is a favorite model among probabilists and statistical physicists for studying critical phenomena. One considers an infinite connected graph $\mathcal{G}$ and takes each edge of $\mathcal{G}$ open or closed with probability p and $1-p$, respectively. All edges are independent of each other. In the simple case this model has just the one parameter p. We write P_p for the corresponding probability measure on configurations of open and closed edges. Let v_0 be some fixed vertex of G and consider the so-called percolation probability $\theta(p) = \theta(p, v_0) := P_p\{$there is an infinite self avoiding open path starting at v_0}, where a path is called open if all its edges are open. If $\mathcal{G} = \mathbb{Z}^d$ or $\mathcal{G}$ is the triangular lattice in the plane, then it was shown by Broadbent and Hammersley that there is some $p_c \in (0,1)$ such that $\theta(p) = 0$ for $p < p_c$ and $\theta(p) > 0$ for $p > p_c$. One can also attach the randomness to the vertices instead of the edges of $\mathcal{G}$. Then the vertices are open or closed with probability p or $1 - p$, respectively. The critical probability is then defined as before. The different critical probabilities are sometimes denoted as $p_c(\mathcal{G}, edge)$ and $p_c(\mathcal{G}, site)$. The study of "critical phenomena" usually refers to the study of singularities of various functions near or at p_c. For instance, let $\mathcal{G}$ be the cluster of v_0, that is the set of all points which can be reached by an open path starting at v_0. Let $|C|$ denote the number of vertices in C, and $E_p|C|$ its expectation. Then it is believed that $E_p|C| \approx (p_c - p)^{-\gamma}$ as $p \uparrow p_c$. Also, at criticality, $P_{p_c}\{|C| \geq n\}$ is expected to behave like $n^{-\rho}$ for some ρ. Constants like γ and ρ are called critical exponents. Physicists not only believe that these exponents exist, but also that they are universal, that is, that they depend only on some simple characteristic of the graph $\mathcal{G}$, such as the dimension. For instance the exponent γ is believed to be the same for the edge and site problems on $\mathbb{Z}^2$ and the edge and site problem on the triangular lattice. This is not the case for the critical probabilities which usually vary with $\mathcal{G}$. In order to explain this universality physicists proposed the "renormalization group", but this has not been made rigorous in the case of percolation. However, this led people to consider whether there existed a scaling limit for critical percolation. More specifically, if one considers percolation at $p_c(\mathcal{G})$ on $\delta\mathcal{G}$ for a periodically imbedded graph $\mathcal{G}$ in $\mathbb{Z}^2$, can one find a limit of the pattern of open paths as $\delta \downarrow 0$? A concrete question is the following: Let D be a "nice" simply connected domain in $\mathbb{C}$ and let A and B be two arcs in ∂D. Let $P_\delta(D, A, B)$ be the probability that in critical percolation on $\delta\mathcal{G}$ there is an open connection in D between A and B. Does $P_0(D, A, B) := \lim_{\delta \downarrow 0} P_\delta(D, A, B)$ exist and what is its value? Here too, it was conjectured that there would be conformal invariance in the limit. For the question at hand that would mean that if $f : D \to D'$ is a conformal homeomorphism from D onto a domain D', then $P_0(D, A, B) = P_0(f(D), f(A), f(B))$. Using conformal invariance and some arguments which have not been made rigorous, John Cardy ([2]) even found explicit values for $P_0(D, A, B)$. Cardy typically took the upper half plane for D and his answers contained various hypergeometric functions. Lennart Carleson realized that the answers would look much simpler when D is an equilateral triangle. This inspired S. Smirnov ([9]) to take the triangular lattice for $\mathcal{G}$ and he succeeded in proving the conformal invariance for this choice.

A connection between critical percolation and SLE is suggested by Schramm in [8]. Schramm writes that assuming a conformal invariance hypothesis, the exploration process should be distributed as SLE_6. The simplest version of the exploration process for critical percolation is as follows: Consider percolation on the hexagonal lattice (the dual of the triangular lattice) and let each hexagonal face be independently open or closed with probability 1/2. Assume that the origin is the center of some hexagon and all hexagons which intersect the open negative real axis (the closed positive real axis) are open (respectively, closed). Then the exploration curve is a curve γ on the boundaries of the hexagonal faces and starting on the boundary of the hexagon containing the origin, such that γ always has an open hexagon on its left and a closed hexagon on its right.

Werner's Work: Power Laws and Critical Exponents for Percolation

Many power laws had been conjectured in the physics literature for the asymptotic behavior of functions at or near the critical point for percolation. In fact, for $\mathcal{G}$ a two-dimensional lattice even the exact value of the corresponding exponents had been predicted. In the two articles [6], [10] Werner and co-authors established most of these power laws and rigorously confirmed the predicted values for the critical exponents in the case of site percolation on the triangular lattice. Before this, no approach which mathematicians deemed rigorous existed for these problems. Even now, the results are restricted to percolation on the triangular lattice because this is the only lattice for which conformal invariance of the scaling limit has been proven.

Previous work, partly by the present author, reduces the problem to the following: Consider critical site percolation on δ times the triangular lattice; this is percolation at $p = 1/2$ for this lattice. Let $b_j(\delta, r, R)$ be the probability that there exist j disjoint crossings of the annulus $\{r \leq |z| < R\}$. Here a crossing may be either an open crossing or a closed crossing, but for $j \geq 2$ we require in b_j that among the j crossings there are some open ones and some closed ones. Then show that $b_j(r, R) = \lim_{\delta \downarrow 0} b_j(\delta, r, R)$ exists and that $\lim_{R \to \infty}[\log R]^{-1} \log b_j(r, R) = -(j^2 - 1)/12$ if $j \geq 2$, and equals 5/48 if $j = 1$. An analogous half-plane problem is to set $a_j(\delta, r, R) =$ the probability that there exist j disjoint open crossings of the semi-annulus $\{z \in \mathbb{H} : r \leq |z| < R\}$. Then show the existence of $a_j(r, R) = \lim_{\delta \downarrow 0} a_j(\delta, r, R)$ and show that $\lim_{R \to \infty}[\log R]^{-1} \log a_j(r, R) = -j(j + 1)/6$. It is much easier to deal with the problem in the half-plane than in the full plane, because it is possible to define the open crossing of the semi-annulus which is closest to the positive real axis. The fact that one cannot single out such a first crossing of the full annulus is also the reason why in b_j with $j \geq 2$ one has to require the existence of both open and closed crossings, but in a_j they can be all of the same type (or, in fact, any mixture of types). The proofs of these limit relations in [6], [10] are based on the result of [9] that the scaling limit for the exploration process in critical percolation on the triangular lattice is an SLE_6 curve. Actually, one needs convergence in a stronger sense than given in [9], but this convergence has now been provided in [1]. The limit relation for percolation is then reduced to finding certain crossing probabilities for SLE_6 which already had been tackled in [4], [5].

Over the years the proofs have been simplified quite a bit. [7] gives a general outline how to prove that a given process has SLE_κ as a scaling limit. One model for which this is still an open problem is self avoiding walk. It is believed that two-dimensional self avoiding walk has $SLE_{8/3}$ as scaling limit. An indication for this is the fact that $SLE_{8/3}$ is the only SLE_κ process which has the so-called restriction property. This property relates SLE conditioned to stay in D with the same SLE process conditioned to stay in D', for a domain D and a subdomain D' (see [12]).

Now that the power of SLE to rigorously analyze some models motivated by physics has been demonstrated, the properties of SLE are also being developed for their own sake (see [3]).

There is a list of open problems in Section 11.2 of Werner's St. Flour Lecture Notes [11]. We may add to this that many of the problems mentioned here can also be formulated in dimension > 2 and the two-dimensional tools of conformal invariance and SLE are probably of little help there.

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