A Metaphor for Mathematics Education

Greg McColm

Most mathematicians don't worry about the philosophy of mathematics. If we have not resolved the crises of the early twentieth century, at least we've learned to live with them. And it's not clear what contemporary philosophers have to say to mathematicians: while philosophy and logic still walk together—perhaps more in parallel than in tandem—and while cognitive scientists collaborate with epistemologists, philosophy seems to offer mathematics in general less than it used to.

Yet philosophy *can* offer self-awareness.

This seems a rather odd commodity for philosophy to offer to mathematics, for presumably we mathematicians are aware of what we're doing. We compose definitions and conjectures, prove theorems and develop algorithms, write papers and teach students, and even serve on committees. Of course we know what we're doing, don't we?

Besides, contemporary philosophy of mathematics doesn't seem to offer self-awareness; its primary concern is what mathematical objects (indeed, set theoretic objects) *are*, if anything. But that's because the philosophers are trying to understand the mathematics we have safely filed away in journals.

But we may need self-awareness. Consider the following sad tale.

In 1844 Hermann Grassman presented his "calculus of extension" in his *Ausdehnungslelire*, which began with an alarmingly philosophical introduction and continued into an algebra which his contemporaries found rather strange. Famous people receiving copies reacted almost characteristically: Gauss said he had done it already, Möbius confessed to being philosophy-phobic and passed it to a friend who apparently never responded, Cauchy seems to have misplaced his copy or gotten confused or something, and Hamilton was favorably but ineffectually impressed. It appears that Grassman's contemporaries understood at best a gist of his algebra; certainly they didn't see what we see in hindsight: an embryonic vector calculus. In 1862 Grassman published a revision of the *Ausdehnungslelire*, even longer, in a more Euclidean format, sans philosophy (but still, according to reviewers, opaque), which also fell flat. Grassman spent a quarter century producing variations of a single difficult and largely unread book, as well as other articles that mathematicians found hard to read, all the while writing patient letters to famous people, most of which seem to have gone unanswered.

Michael Crowe[1] suggests that Grassman's problem may have been his lack of students and credentials (he studied philology and theology in Berlin and taught at technical schools, but never got a university post) and the novelty of his approach. And there was the opacity of his exposition. Put another way, he did not persuade his contemporaries to make the effort to follow up on his work, and he did not develop a navigable path for his contemporaries to follow to his discovery: his problems were essentially *pedagogical*.

Here is where philosophy might help: a "navigable path"? Through what? And to what?

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[1] *This account from M. J. Crowe,* A History of the Vector Calculus: The Evolution of the Idea of a Vectorial System, *Dover, rev. ed., 1994. Crowe's suggestion is on pp. 94–95.*

Mathematicians tend to act as if we discover things more than inventing them. We behave as if there really is, out there, somewhere, that celebrated sponsor of many episodes of *Sesame Street*, The Number Seven. This *Platonic* view is that mathematics is somehow generated by abstract ideas or "forms", like The Number Seven. On the other hand, the *Aristotelian* view is that forms do not generate objects, but rather the other way around: (our knowledge of) mathematical objects somehow arise from observing phenomena. These views are two poles of a vast, wide spectrum of philosophies, but Plato and Aristotle still dominate the landscape, with philosophers tending to lean one way or the other.

After two millennia, there is still no consensus on whether linear operators on Hilbert spaces are as real as lions in the Serengeti. But there is a compromise: while what is "real" may be unknown or even unknowable, our perceptions and our thoughts about our perceptions are knowable, and that is enough for science[2]—at least, for a science that concerns itself with explanation rather than "truth".

Thus Hilbert says that while we might not be able to visualize the "cardinality of the continuum", we can engineer a nomenclature that allows us to manipulate it.[3] We see this as engineering because of its quirks, e.g., when a notion doesn't point at the object we thought it pointed at, when several quite different notions point at the same object, or simply when our nomenclature trips over itself.[4] The mathematics in our literature is our own creation, distinct from the "real" mathematics "out there" that our literature is "about".

We propose a metaphor capturing this distinction.

Imagine a plain on which a vast, invisible edifice supposedly rises up to the sky. We know it's there because of its effect on the plain and the climate around it. People plant near what seems to be the base of the edifice the seeds of a vine that grows up the invisible walls, feeling its way along the nooks, crannies, and statuary, slowly producing an outline of... something there. This vine does not grow at will, for it needs constant care. It needs water and fertilizer and even directing, hence gardeners. These caretakers constantly prune and poke at it, from the ground or while standing on totteringly tall ladders. The result is that the shape of the vine and the outline it appears to make reflect not only the edifice within but also the interests and agendas of the gardeners.

The vine forms a history of a dialogue with the universe. X explores a line of thought, Y conducts an experiment, and Z writes a book. When X comes up with some new results, what her colleagues see is not what X glimpsed through a glass darkly, but what she wrote on paper. When Y applies those results in his laboratory, he is using something whose origin is part edifice, part vine. And then Z, squatting among stacks of papers she is trying to resolve into a coherent work, deals regularly with the paper trails of darkly glimpsed phenomena and so conducts a dialogue of her own. All three produce the substance of the vine.

There are three ways that the vine grows. The vine grows extensively: tendrils explore the way, pushing forward, upward, and inward. This is the frontier growth that we read about in newspapers and history books. The vine grows expansively: the branches may form an outline of a turret, but tendrils still poke around, finding more patterns and shapes on the surface (and occasional passageways), filling in the spaces between the major initial discoveries in a field. This is the "mature science" of Thomas Kuhn.[5] And the vine grows intensively, reorganizing its structure by merging branches to form great boughs (the abstractions made of many smaller preceding theories), producing the mathematics of synthesizers, expositors, and teachers.

We are the caretakers. If we see ourselves as the creators and maintainers of this ancient cultural creation I call a vine, which we use indirectly to study a reality we may not directly apprehend, then we can see that by understanding the philosophical (and sociological!) problem of how the vine behaves, we are in a better position to do our jobs. The philosophers may have something critical to say to us after all.

Let's turn to pedagogy.

Grassman's predicament is a familiar one to many teachers. It's all so clear, but the students can't or won't follow. In fact, introducing something new to one's students is a bit like introducing something new to one's colleagues. Imagine that you are one of several caretakers, tending a vine as it sends tendrils creeping up an invisible wall. This is the sort of research that many of us do: accompanied by five to fifty colleagues worldwide, we explore a section of the wall. We report our

[2] *The philosopher most strongly associated with this sort of view is Immanuel Kant: see his* Prolegomena to Any Future Metaphysics, *which is slightly more accessible and a lot shorter than his* Critique of Pure Reason. *For a more clear, plausible, and moderate distillation, see A. F. Chalmers,* What Is This Thing Called Science?, *Univ. of Queensland Press, 2nd ed., 1976.*

[3] *D. Hilbert, "On the infinite", reprinted in* Philosophy of Mathematics *(P. Benacerraf and H. Putnam, eds.), Cambridge Univ. Press, 1983, pp. 183–201.*

[4] *See I. Lakatos,* Proofs and Refutations: The Logic of Mathematical Discovery, *(J. Worrall and E. Zahar, eds.), Cambridge Univ. Press, 1976, for examples.*

[5] *T. Kuhn,* Structure of Scientific Revolutions, *Univ. of Chicago Press, 2nd ed., 1970.*

results to journals that specialize in the upper north-northwestern face, although the journals on the westernmost of the northern turrets have been known to take our papers. A vertical mass of vine on the north-northwestern section of the edifice appears to be growing up a wall, and the clump to the west seems to be growing up some kind of cylinder. After many such assessments, the American Mathematical Society has divided up the sections of the vine as if that division reflects the architecture of the edifice itself, and the National Science Foundation divides its funding according to fashion and probable economic return associated with various sections of the vine.

But suppose your tendril encounters something unexpected, like an open space amidst some projections. A barred passageway? Exhilarated but anxious, you explore this strange opening. You write an article about it, and then... the first journal thinks you dropped a minus sign, the second thinks you've lost your marbles, and the third journal thinks your results aren't "interesting". As your champagne goes flat, you may wonder what's wrong with everybody.

Here's what the referees think. Perhaps your tendrils lost hold of the edifice and are now growing into a pointless tangle. Or perhaps there is something there, but they cannot navigate the snarl. (And unless there is some kind of authority behind it, mathematicians tend to be wary of muddle: like our students, we want reassurance—perhaps credentials printed on parchment—of the competence of our guide before we invest hours of our time.) In order to lead one's colleagues to something new, they have to be led via familiar-looking paths. There is a theory of "processing fluency"[6] that suggests that both the difficulty of exposition and the apparent unfamiliarity of the topic may undermine the (perceived) credibility of the thesis.

Grassman found the turret that everybody wanted (how to do calculus in Euclidean space), but the approach was difficult, and Grassman never figured out how to lead anyone there. Later, when the vine had grown up closer to the turret and there were more places to approach this turret (from Gibbs's mechanics to Heaviside's electricity), that place was discovered by two people who more effectively described the path and sold the idea of visiting the place—especially Willard Gibbs, who bombarded his colleagues with fusillades of preprints. So thanks more to Gibbs and Heaviside than to Grassman, vector calculus is now entrenched in the curriculum.

Since we can only see the vine, our colleagues can only follow the green we have grown for them, down the paths we have made. The problem is pedagogical, and if there is an ignored elephant in Western philosophy's living room, it is pedagogy.[7] Much mathematical "research" consists of teaching mathematics to our colleagues. Part of Grassman's problem was that while he did find a route to the turret, he did not develop his bit of vine into a stem that led cleanly up it.

Discovery only generates unorganized reams of reports of trailblazing here and there. Science is a social activity, so there is more to science than discovery. There is also organization. There are plenary talks, expository articles, advanced topics courses, multiyear projects culminating in retrospective research monographs, followed by graduate texts. Visitors from other fields pick up nuggets to be baked in academic kilns and presented, in spruced up or dumbed down form, for strange applications the original discoverers never imagined. And then it's on to the massive apparatus of undergraduate education.

The undergraduate curriculum is not at all like the confusing tangle of the canopy. Students start at ground level, where the underbrush has been cleared away and where the base of the vine is an orderly array of well-tended trunks, with helpful and well-tended limbs ("examples" and "exercises") to assist climbing novices. Our students slowly work their way up from the park we maintain for beginners, and slowly the problems get more complicated, less well defined, with more stray branches (or are they stray branches?) and so on, until finally they are just like us.

Consider one of the conundrums of first year calculus: what do we do about limits? We can skip past the rigors of ϵ and δ, but this seems to deprive or even frustrate our more curious and demanding students.[8] So traditionally we had our students stretch towards the ϵ-δ branch in Calculus I, and then towards an ϵ-N branch in Calculus II, and then back towards ϵ-δ in Calculus III, so that after all this yoga they can actually grasp the branch in Advanced Calculus.

But many students flail and give up, so most calculus classes have dropped the epsilons. Some teachers see an alternative route using Abraham Robinson's nonstandard analysis, with some reports of success.[9] Whatever one's position on the debate over epsilons versus infinitesimals, we do

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[6] *See R. Reber, N. Schwarz and P. Winkielman, "Processing fluency and aesthetic pleasure: Is beauty in the perceiver's processing experience?",* Personality and Social Psychology Review ***8****:4 (2004), 364–382.*

[7] *Eastern philosophy, on the other hand, has devoted a great deal of attention to pedagogy.*

[8] *J. E. Szydlik, "Mathematical beliefs and conceptual understanding of the limit of a function",* J. Res. Math. Ed. ***13****:3 (2000), 258–276.*

[9] *K. Sullivan, "The teaching of elementary calculus using the nonstandard analysis approach",* Amer. Math. Monthly ***85****:5 (1976), 370–375.*

see here the arboreal aspect of mathematics.[10] Out of historical accident[11] two vine branches embrace the same invisible rampart, and now we debate which vine is the more intelligible, the more powerful, the more practical, etc. (Some combatants even debate which is more like the invisible rampart, as if that was a resoluble issue.) Both growths are human creations, the results of our endeavor to navigate and make out this difficult but apparently critical invisible structure, and ultimately the debate is over the utility of each part of vine in accomplishing this navigation and perception.

Like our students, we prefer to stand on an orderly array of well-tended boughs; even the Zen masters of complexity display their talents by cleaning up the messes they encounter; even the great and messy problem solvers need a safe place to stand. And if each researcher sees her or his own array of branches, an outsider often just sees—a tangle. If we are to persuade outsiders to join us, we have to remember our audience and design our section of vine to entice visitors. Mathematical truth—whatever that is—is not what we deal with daily; it is our mathematical gardening that we share. And like all human productions, mathematical gardening require users' manuals, packaging, and salesmanship.

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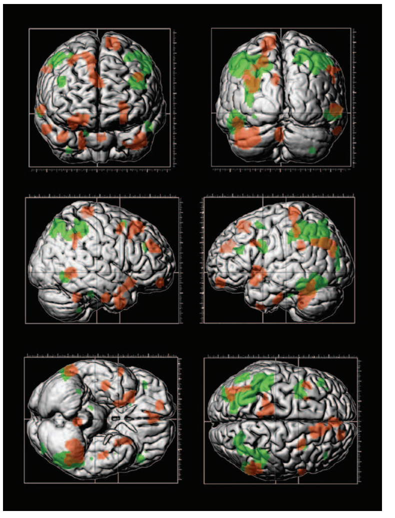
[10] *Hilbert, remember, would call this an engineering aspect.*

[11] *See, e.g., I. Kleiner, "History of the infinitely small and the infinitely large in calculus",* Ed. Stud. Math. ***48*** *(2001), 137–174.*

About the Cover



Mathematical exercises

April is Mathematics Awareness Month, and this year's theme is the connection between mathematics and the brain. The webpage is at `http://www.mathaware.org/mam/` and there ought to be some interesting short essays posted there. What those who chose this theme presumably had in mind was the sophisticated mathematics that has gone into the technology that analyzes brain functions. But also of interest is the possibility that analyis of brain activity can tell us something about how humans do mathematics. In his book on mathematical invention, the eminent French mathematician Jacques Hadamard asked, "Will it ever happen that mathematicians know enough of that subject of the physiology of the brain and that neurophysiologists know enough of mathematical discovery for efficient cooperation to be possible?" The answer seems now "Very likely".

Mauro Pesenti and colleagues have studied thoroughly the brain activity of a calculating prodigy named Rüdiger Gamm, and it is his brain displayed on the cover of this month's issue. The sorts of things he does remarkably well in his head include multi-digit multiplication, computation of sines, and calendrical calculations. The areas of the brain displayed in green are those used by both Gamm and nonexpert control subjects while doing mental arithmetic, and those in red are those used only by Gamm. What is interesting is that the areas used only by Gamm are generally those associated to episodic long-term memory,

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