WHAT IS... a Tropical Curve?

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A tropical curve is an algebraic curve defined over the semifield $\mathbb{T}$ of tropical numbers. The goal of this note is to make sense out of this phrase.

Figure 1 depicts a union of two planar tropical curves (also known in physics as (p, q)-webs), namely a tropical conic and a tropical line. Each of them may look very different from its classical counterpart, but they do share many features, e.g., a line (the tripod graph in the lower part of the picture) intersects a conic (the rest of the picture) in two points. In can be shown that tropical curves come as limits of classical curves (Riemann surfaces) under a certain procedure degenerating their complex structure. Tropical curves proved to be useful *en lieu* of honest holomorphic curves in a range of classical problems.

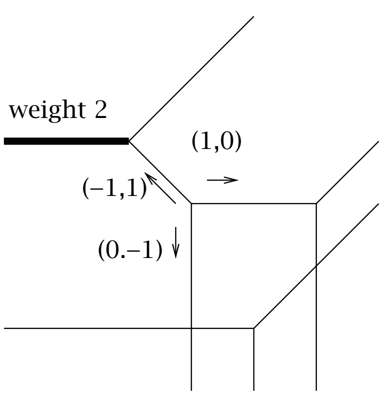

Figure 1. A conic and a line in the plane.

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We define the tropical semifield $\mathbb{T}$ to be the set $\mathbb{R} \cup \{-\infty\}$, and we equip it with the *"addition"* operation "$x + y$" $= \max\{x, y\}$ and with the "multiplication" operation "xy" $= x + y$. We use the quotation marks to distinguish between the standard and tropical arithmetic operations. Our "additive zero" is $-\infty$ while the "multiplicative unit" is 0. We have $\mathbb{T}^\times = \mathbb{T} \setminus \{-\infty\} = \mathbb{R}$.

The term *tropical* is taken from computer science, where it was coined to commemorate contributions of the Brazilian school. The term *semifield* refers to the properties of the tropical arithmetic operations: we have all the axioms required for a *field*, except for the existence of subtraction (as our addition is idempotent "$x + x$" $= x$). Luckily, one does not need subtraction to write down polynomials (they are *sums* of monomials)!

Consider a polynomial in two variables

$$p(x, y) = \text{“}\sum_{j,k} a_{jk} x^j y^k\text{”} = \max_{j,k}(jx + ky + a_{jk}).$$

The tropical curve C defined by p consists of those points $(x, y) \in \mathbb{R}^2$ where p is not differentiable. In other words, C is the locus where the maximum is assumed by more than one of the "monomials" of p. It is easy to see that $C \subset \mathbb{R}^2$ is a graph and its edges are straight intervals with rational slopes.

The edge E, where "$a_{j_1k_1} x^{j_1} y^{k_1}$" $=$ "$a_{j_2k_2} x^{j_2} y^{k_2}$", is perpendicular to the vector $(j_1 - j_2, k_1 - k_2)$. We can enhance E with a natural number $w(E)$ (called its *weight*) equal to $\mathrm{GCD}(j_1 - j_2, k_1 - k_2)$.

Take a vertex $A \in C$ and consider the edges $E_1, \ldots, E_n$ adjacent to A. Let $v(E_j) \in \mathbb{Z}^2$ be the primitive integer vector from A in the direction of E_j. It is easy to see that we have the following *balancing* (or zero-tension) condition at each vertex of C:

$$\sum_{j=1}^{n} w(E_j)v(E_j) = 0. \tag{1}$$

Furthermore, one can easily show that any weighted piecewise-linear graph in $\mathbb{R}^2$ with rational slopes of the edges and with the zero-tension condition at the vertices is given by some tropical polynomial.

The plane $\mathbb{R}^2$ can be thought of as a part of the tropical affine plane $\mathbb{T}^2 = [-\infty, +\infty)^2$. The regular functions on $\mathbb{T}^2$ are tropical polynomials, and the regular functions on $\mathbb{R}^2 = (\mathbb{T}^\times)^2$ are tropical Laurent polynomials. Note that a monomial is an affine-linear function with an integer slope and therefore the geometric structure on $\mathbb{R}^2$ encoding the tropical structure is the $\mathbb{Z}$*-affine structure*.

The plane $\mathbb{T}^2$ can be compactified to the projective plane $\mathbb{TP}^2$. To construct $\mathbb{TP}^2$ we take the quotient of $\mathbb{T}^3 \setminus \{(-\infty, -\infty, -\infty)\}$ by the usual equivalence relation $(x, y, z) \sim (\lambda x, \lambda y, \lambda z)$, $\lambda \neq 0$. As in the classical case we have three affine charts, so $\mathbb{TP}^2$ can be obtained by gluing three copies of $\mathbb{T}^2$. Thus we may think of $\mathbb{TP}^2$ as a triangle-like compactification of $\mathbb{R}^2$ taken with the tautological $\mathbb{Z}$-affine structure. Each side of the triangle corresponds to a copy of $\mathbb{TP}^1$ (which is itself a compactification of $\mathbb{R}$ by two points). Similarly, we may define $\mathbb{TP}^n \supset \mathbb{R}^n$ as well as other tropical toric varieties.

We have compact tropical curves in $\mathbb{TP}^n$. Let Γ be a finite graph and $h : \Gamma \to \mathbb{TP}^n$ be a continuous map that takes the interior of every edge E to a straight (possibly unbounded) interval with a rational slope in $\mathbb{R}^n$. If we can prescribe a positive integer weight to each edge so that (1) holds at every vertex in $\mathbb{R}^n$, then we say that $h : \Gamma \to \mathbb{TP}^n$ is a tropical curve.

The *degree* d of $h(\Gamma)$ is the intersection number with any of the $(n+1)$ $\mathbb{TP}^{n-1}$-divisors at infinity. The degree can be calculated by examining the unbounded edges $E_1, \dots, E_l$. For example the intersections with the "last" infinity divisor $D_\infty = \mathbb{T}P^n \setminus \mathbb{T}^n$ are given by those E_j whose outward primitive vectors $v(E_j)$ have at least one coordinate positive; the local intersection number with D_∞ is $w(E_j)$ times that coordinate (assuming that it is maximal), and d is the sum of these local intersection numbers. The balancing condition ensures that the total intersection number with all infinity divisors is the same. The *genus* of Γ is $g = \dim H_1(\Gamma)$. We see that both the line and the conic from Figure 1 have genus 0.

Note that tropical curves behave quite similarly to classical algebraic curves. Prove (as an exercise) that any two points in $\mathbb{TP}^n$ can be connected with a line. Curves in $\mathbb{TP}^n$ of degree d and genus g vary in a family of dimension at least $(n+1)d + (n-3)(1-g)$. In many cases this lower bound is exact, for instance if $g = 0$ (for any n) or if $n = 2$ (for any g) if h is an immersion.

For example if we fix a configuration C of $3d - 1 + g$ generic points in $\mathbb{TP}^2$ then only finitely many curves $h_j : \Gamma_j \to \mathbb{TP}^2$ of degree d and genus g will pass through C. In contrast to the case of complex coefficients, the actual number of such tropical curves h_j will depend on the choice of C. However, each such tropical curve comes with a combinatorial multiplicity so that the number of curves with multiplicity is invariant. And this invariant coincides with the number of complex curves of degree d and genus g passing through a generic configuration of $3d - 1 + g$ points in $\mathbb{CP}^2$ and gives an efficient way of computing that number.

There is also a different choice of multiplicities for h_j, responsible for enumeration over $\mathbb{R}$ (some real curves are counted with the sign $+1$ and some with -1, and their total number is different from the complex counterpart). The sum of the real multiplicities of $h_j(\Gamma_j)$ gives the answer for the corresponding real enumerative problem.

As an example consider the case $d = 3$ and $g = 0$ (see Figure 2). We fix a configuration C of 8 points in $\mathbb{R}^2 \subset \mathbb{TP}^2$. Depending on the choice of C there might be 9 or 10 tropical curves via C. However, the sum of their complex multiplicities is always 12 while the sum of their real multiplicities is always 8.

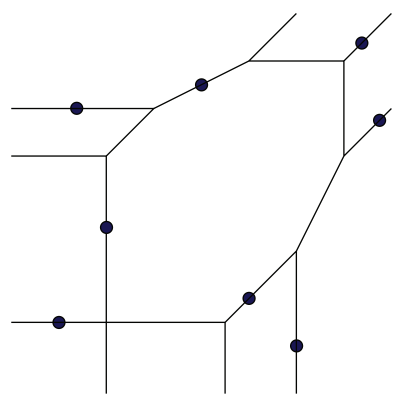

Figure 2. A cubic of multiplicity 1 via 8 points.

A map $h : \Gamma \to \mathbb{TP}^n$ can be used to induce tropical structure on Γ. Tropical monomials p in $\mathbb{R}^n = (\mathbb{T}^\times)^n$ give smooth functions on every edge $E \subset \Gamma$ and can be used to measure the length of E. Define the length of E to be the smallest of such lengths divided by the weight of E. This turns Γ into a metric graph: the leaves (i.e., the edges adjacent to 1-valent vertices) get infinite length while the inner edges are finite.

Conversely, a metric graph structure on Γ can be used to define *tropical* maps $\Gamma \to X$, where X is $\mathbb{T}^n$, $\mathbb{TP}^n$, or any other tropical variety (which can be defined in higher dimension as a polyhedral complex equipped with an integer affine structure; only in dimension 1 we can hide the integer affine structure under the guise of a metric). Higher weight appears when h "stretches" the edges by an integer amount.

There is an equivalence relation between tropical curves generated by the following relation: at any point $x \in \Gamma$ we may introduce an infinite length interval connecting x with a new 1-valent vertex. This equivalence allows us to turn a map given by regular functions into a tropical morphism. Also it allows us to treat any marked point as a 1-valent vertex. This turns, e.g., the space $\mathcal{M}_{0,n}$ of trees with n marked points into an $(n-3)$-dimensional tropical variety.

Most classical theorems on Riemann surfaces have counterparts for tropical curves, in particular, the Abel-Jacobi theorem, the Riemann-Roch theorem, and the Riemann theorem on the θ-functions. Many features of complex and real curves become easily visible after tropicalization.

Further reading.

[1] G. Mikhalkin, Enumerative tropical algebraic geometry in $\mathbb{R}^2$, *J. Amer. Math. Soc.* **18** (2005), no. 2, 313–377.

[2] G. Mikhalkin and I. Zharkov, Tropical curves, their Jacobians and Theta-functions, http://arxiv.org/abs/math.AG/0612267.

[3] J. Richter-Gebert, B. Sturmfels, and T. Theobald, *First steps in tropical geometry*, Contemporary Mathematics, vol. 377, Amer. Math. Soc., 2005, pp. 289–317.