A tropical curve is an algebraic curve defined over the semifield \( T \) of tropical numbers. The goal of this note is to make sense out of this phrase.

Figure 1 depicts a union of two planar tropical curves (also known in physics as \( (p,q) \)-webs), namely a tropical conic and a tropical line. Each of them may look very different from its classical counterpart, but they do share many features, e.g., a line (the tripod graph in the lower part of the picture) intersects a conic (the rest of the picture) in two points. It can be shown that tropical curves come as limits of classical curves (Riemann surfaces) under a certain procedure degenerating their complex structure. Tropical curves proved to be useful en lieu of honest holomorphic curves in a range of classical problems.

We define the tropical semifield \( T \) to be the set \( \mathbb{R} \cup \{ -\infty \} \), and we equip it with the “addition” operation \( x + y = \max \{ x, y \} \) and with the “multiplication” operation \( xy = x + y \). We use the quotation marks to distinguish between the standard and tropical arithmetic operations. Our “additive zero” is \(-\infty\) while the “multiplicative unit” is 0. We have \( T^\times = T \setminus \{-\infty\} = \mathbb{R} \).

The term tropical is taken from computer science, where it was coined to commemorate contributions of the Brazilian school. The term semifield refers to the properties of the tropical arithmetic operations: we have all the axioms required for a field, except for the existence of subtraction (as our addition is idempotent \( x + x = x \)). Luckily, one does not need subtraction to write down polynomials (they are sums of monomials)!

Consider a polynomial in two variables

\[
p(x, y) = \sum_{j,k} a_{jk} x^j y^k = \max_{j,k} (jx + ky + a_{jk}).
\]

The tropical curve \( C \) defined by \( p \) consists of those points \( (x, y) \in \mathbb{R}^2 \) where \( p \) is not differentiable. In other words, \( C \) is the locus where the maximum is assumed by more than one of the “monomials” of \( p \). It is easy to see that \( C \subset \mathbb{R}^2 \) is a graph and its edges are straight intervals with rational slopes.

The edge \( E \), where \( a_{j_1 k_1} x^{j_1} y^{k_1} = a_{j_2 k_2} x^{j_2} y^{k_2} \), is perpendicular to the vector \( (j_1 - j_2, k_1 - k_2) \). We can enhance \( E \) with a natural number \( w(E) \) (called its weight) equal to \( \gcd(j_1 - j_2, k_1 - k_2) \).

Take a vertex \( A \in C \) and consider the edges \( E_1, \ldots, E_n \) adjacent to \( A \). Let \( v(E_j) \in \mathbb{Z}^2 \) be the primitive integer vector from \( A \) in the direction of \( E_j \). It is easy to see that we have the following balancing (or zero-tension) condition at each vertex.
of $C$:

$$\sum_{j=1}^{n} w(E_j)v(E_j) = 0. \tag{1}$$

Furthermore, one can easily show that any weighted piecewise-linear graph in $\mathbb{R}^2$ with rational slopes of the edges and with the zero-tension condition at the vertices is given by some tropical polynomial.

The plane $\mathbb{R}^2$ can be thought of as a part of the tropical affine plane $\mathbb{T}^2 = (-\infty, +\infty)^2$. The regular functions on $\mathbb{T}^2$ are tropical polynomials, and the regular functions on $\mathbb{R}^2 = (\mathbb{T}^*)^2$ are tropical Laurent polynomials. Note that a monomial is an affine-linear function with an integer slope and therefore the geometric structure on $\mathbb{R}^2$ encoding the tropical structure is the $\mathbb{Z}$-affine structure.

The plane $\mathbb{T}^2$ can be compactified to the projective plane $\mathbb{T}\mathbb{P}^2$. To construct $\mathbb{T}\mathbb{P}^2$ we take the quotient of $\mathbb{T}^3 \setminus \{(-\infty, -\infty, -\infty)\}$ by the usual equivalence relation $(x, y, z) \sim (\lambda x, \lambda y, \lambda z), \lambda \neq 0$. As in the classical case we have three affine charts, so $\mathbb{T}\mathbb{P}^2$ can be obtained by gluing three copies of $\mathbb{T}^2$. Thus we may think of $\mathbb{T}\mathbb{P}^2$ as a triangle-like compactification of $\mathbb{R}^2$ taken with the tautological $\mathbb{Z}$-affine structure. Each side of the triangle corresponds to a copy of $\mathbb{T}\mathbb{P}^1$ (which is itself a compactification of $\mathbb{R}$ by two points). Similarly, we may define $\mathbb{T}\mathbb{P}^n \supset \mathbb{R}^n$ as well as other tropical toric varieties.

We have compact tropical curves in $\mathbb{T}\mathbb{P}^n$. Let $\Gamma$ be a finite graph and $h : \Gamma \to \mathbb{T}\mathbb{P}^n$ be a continuous map that takes the interior of every edge $E$ to a straight (possibly unbounded) interval with a rational slope in $\mathbb{R}^n$. If we can prescribe a positive integer weight to each edge so that (1) holds at every vertex in $\mathbb{R}^n$, then we say that $h : \Gamma \to \mathbb{T}\mathbb{P}^n$ is a tropical curve.

The degree $d$ of $h(\Gamma)$ is the intersection number with any of the $(n+1)$ $\mathbb{T}\mathbb{P}^{n-1}$-divisors at infinity. The degree can be calculated by examining the unbounded edges $E_1, \ldots, E_i$. For example the intersections with the “last” infinity divisor $D_\infty = \mathbb{T}\mathbb{P}^n \setminus \mathbb{T}^n$ are given by those $E_j$ whose outward primitive vectors $v(E_j)$ have at least one coordinate positive; the local intersection number with $D_\infty$ is $w(E_j)$ times that coordinate (assuming that it is maximal), and $d$ is the sum of these local intersection numbers. The balancing condition ensures that the total intersection number with all infinity divisors is the same. The genus of $\Gamma$ is $g = \dim H_1(\Gamma)$. We see that both the line and the conic from Figure 1 have genus 0.

Note that tropical curves behave quite similarly to classical algebraic curves. Prove (as an exercise) that any two points in $\mathbb{T}\mathbb{P}^n$ can be connected with a line. Curves in $\mathbb{T}\mathbb{P}^n$ of degree $d$ and genus $g$ vary in a family of dimension at least $(n+1)d + (n-3)(1-g)$. In many cases this lower bound is exact, for instance if $g = 0$ (for any $n$) or if $n = 2$ (for any $g$) if $h$ is an immersion.

For example if we fix a configuration $C$ of $3d - 1 + g$ generic points in $\mathbb{T}\mathbb{P}^2$ then only finitely many curves $h_j : \Gamma_j \to \mathbb{T}\mathbb{P}^2$ of degree $d$ and genus $g$ will pass through $C$. In contrast to the case of complex coefficients, the actual number of such tropical curves $h_j$ will depend on the choice of $C$. However, each such tropical curve comes with a combinatorial multiplicity so that the number of curves with multiplicity is invariant. And this invariant coincides with the number of complex curves of degree $d$ and genus $g$ passing through a generic configuration of $3d - 1 + g$ points in $\mathbb{C}\mathbb{P}^2$ and gives an efficient way of computing that number.

There is also a different choice of multiplicities for $h_j$, responsible for enumeration over $\mathbb{R}$ (some real curves are counted with the sign +1 and some with -1, and their total number is different from the complex counterpart). The sum of the real multiplicities of $h_j(\Gamma_j)$ gives the answer for the corresponding real enumerative problem.

As an example consider the case $d = 3$ and $g = 0$ (see Figure 2). We fix a configuration $C$ of 8 points in $\mathbb{R}^2 \subset \mathbb{T}\mathbb{P}^2$. Depending on the choice of $C$ there might be 9 or 10 tropical curves via $C$. However, the sum of their complex multiplicities is always 12 while the sum of their real multiplicities is always 8.

![Figure 2. A cubic of multiplicity 1 via 8 points.](image-url)
be defined in higher dimension as a polyhedral complex equipped with an integer affine structure; only in dimension 1 we can hide the integer affine structure under the guise of a metric). Higher weight appears when \( h \) "stretches" the edges by an integer amount.

There is an equivalence relation between tropical curves generated by the following relation: at any point \( x \in \Gamma \) we may introduce an infinite length interval connecting \( x \) with a new 1-valent vertex. This equivalence allows us to turn a map given by regular functions into a tropical morphism. Also it allows us to treat any marked point as a 1-valent vertex. This turns, e.g., the space \( \mathcal{M}_{0,n} \) of trees with \( n \) marked points into an \( (n - 3) \)-dimensional tropical variety.

Most classical theorems on Riemann surfaces have counterparts for tropical curves, in particular, the Abel-Jacobi theorem, the Riemann-Roch theorem, and the Riemann theorem on the \( \theta \)-functions. Many features of complex and real curves become easily visible after tropicalization.

**Further reading.**


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